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Abstract

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ON THE DISCRETE PART OF THE SPECTRUM OF THE LAPLACIAN IN LIMITING CYLINDRICAL DOMAINS

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Restricting ourselves, for definiteness, to three independent variables, we shall call an infinite domain Ω of three-dimensional Euclidean space **limiting-cylindrical** if, in some cylindrical coordinate system (r, φ, z), it is bounded by the surface

$$r = r(\varphi)[1 + \delta(z)] \quad (0 \leq \varphi < 2\pi,\ \delta(z) > -1,\ 0 \leq z < \infty), \tag{1}$$

where

$$\lim_{z \to \infty} \delta(z) = 0, \tag{2}$$

and by the finite part of the plane $z = 0$ cut off by this surface (1). The function $r(\varphi)$ is assumed to be continuous and 2π-periodic. By $-\widetilde{\Delta}$ we denote the self-adjoint operator, defined in $\mathcal{L}_2(\Omega)$, of the Laplacian $-\Delta$ with zero boundary condition on the boundary of the domain Ω. Further, let α be the first eigenvalue of the boundary-value problem defined by the two-dimensional Laplacian $-\Delta$ in the part of the plane $z = 0$ bounded by the curve

$$r = r(\varphi) \quad (0 \leq \varphi < 2\pi), \tag{3}$$

with zero condition on the boundary.

As is known ([1]), to the left of the point α the spectrum $S(-\widetilde{\Delta})$ will be discrete. On the other hand, it is easy to show that the point α itself already belongs to the continuous part of the spectrum $C(-\widetilde{\Delta})$.

Thus, *a priori* there are two possibilities: either the continuous part of the spectrum of $-\widetilde{\Delta}$ is preceded by only a finite number of its eigenvalues (property f), or it is preceded by an infinite set of its eigenvalues (property af).

In accordance with (1) and (2), the limiting cylindrical domain Ω is formed by means of an infinitesimal at infinity perturbation of the cylindrical domain Z with base $z = 0$, directrix (3), and generators parallel to the z-axis. If this perturbation is finite (i.e., outside a finite interval $\delta(z) = 0$), then ([1]) property f holds.

The aim of the present note is to find conditions to which, generally speaking, a non-finite perturbation $\delta(z)$ should be subjected in order to ensure property f or property af, and also to study the asymptotic distribution of the eigenvalues of the operator $-\widetilde{\Delta}$ in the left half-neighborhood of the point α. The final part of the note is devoted to certain additional problems arising in connection with the questions indicated.

The proof of all the results given below is based on the splitting principle ([2]) and on P. Courant' s variational principles. In a certain sense the present note adjoins the investigations of F. Rellich ([3]) and D. Jones ([1]).

The notation used above retains its meaning throughout the remainder of the note.

1. The following analogue of the well-known oscillation theorem of A. Kneser holds.

Theorem 1. If

$$\limsup_{z\to\infty} z^2\delta(z) < \frac{1}{8\alpha}, \tag{4}$$

then to the left of α there can lie only a finite number of eigenvalues of the operator $-\widetilde{\Delta}$.

If

$$\liminf z^2\delta(z) > \frac{1}{8\alpha}, \tag{5}$$

then the part of the spectrum of the operator $-\widetilde{\Delta}$ lying to the left of the point α consists of an infinite sequence of eigenvalues with the unique limit point $\lambda = \alpha$.

The **proof** of the theorem, by the known method [4], is reduced to the study of the sign of the quadratic functional

$$\Phi[u] = \iiint\limits_{(\Omega)} (|\nabla u|^2 - \alpha|u|^2)\, dx\, dy\, dz. \tag{6}$$

Replacing in (6) the variables according to the formulas $x = [1 + \delta(z)]\xi$, $y = [1+\delta(z)]\eta$, $z = z$, and setting $u(x, y, z) = v(\xi, \eta, z)$, it is not difficult to obtain the double inequality $F_1[v] \leqslant \Phi[u] \leqslant F_2[v]$, where $F_{1,2}[v] = (l_{1,2}v, v)$ are quadratic functionals in $\mathcal{L}_2(\widetilde{Z})$ such that the corresponding differential equations $l_{1,2}[v] = \lambda v$ admit separation of variables $v(\xi, \eta, z) = w(\xi, \eta)\zeta(z)$. After this, the study of the sign of the functionals $F_{1,2}$ of functions of three variables is reduced to the study of the sign of the corresponding quadratic functionals of the function $\zeta(z)$ in $\mathcal{L}_2(0, \infty)$. Replacing, in this study, the function $\delta(z)$, for some $\beta > 0$ and large z, by $\mu_{1,2} = \dfrac{1 \mp \beta}{8\alpha z^2}$ and extending the functions $\mu_{1,2}(z)$ in a suitable manner to the point $z = 0$, we obtain a complete proof of Theorem 1.

2. If the perturbation $\delta(z)$ satisfies condition (5), then, obviously, the number $m(\varepsilon)$ of eigenvalues of the operator $-\widetilde{\Delta}$ that are less than $\alpha - \varepsilon$ will, as $\varepsilon \downarrow 0$, grow the faster, the more slowly the function $\delta(z)$ decreases as $z \to \infty$. To obtain the corresponding asymptotic formula, we shall assume the function $\delta(z)$ to be positive and monotonically decreasing. For small $\varepsilon > 0$, denote by ω_ε the finite part of the domain Ω cut off from it by the plane $z = a(\varepsilon)$, where the number $a(\varepsilon)$ is such that the first eigenvalue of the boundary-value problem for the two-dimensional Laplacian $-\Delta$ in the section $z = a(\varepsilon)$, with zero boundary condition on its boundary, is equal to $\alpha - \varepsilon$. Next, denote by $k(\varepsilon)$ the number of eigenvalues of the operator $-\Delta$ in $\mathcal{L}_2(\omega_\varepsilon)$, with zero boundary condition on the boundary ω_ε, which are less than $\alpha - \varepsilon$. By $k'(\varepsilon)$ denote the number of eigenvalues less than $\alpha - \varepsilon$ of the boundary-value problem obtained from the preceding one by replacing the zero boundary condition on the boundary $z = a(\varepsilon)$ by the condition $\partial u/\partial n = 0$. Then [1]

$$k(\varepsilon) < m(\varepsilon) < k'(\varepsilon). \tag{7}$$

Now dividing the interval $0 \leqslant z \leqslant a(\varepsilon)$ into equal parts by the points $z_0 = 0, z_1, z_2, \dots, z_{n-1}, z_n = a(\varepsilon)$, we form, from cylinders of heights $z_{i+1} - z_i$ $(i = 0, 1, \dots, n{-}1)$, inscribed in and, respectively, circumscribed about the domain ω_ε, stepwise bodies Λ_ε and Λ'_ε. Next, introduce into consideration two boundary-value problems defined by the operation $-\Delta$ in the domains Λ_ε and Λ'_ε. Here, on the cylindrical portions of the boundary of these domains

we impose zero conditions, and on the end partitions $z = z_i$ $(i = 0, 1, \dots, n)$ in the first case the condition $u = 0$, and in the second $\partial u/\partial n = 0$. From R. Courant' s variational principles there follow, for the numbers $v(\varepsilon)$ and $v'(\varepsilon)$ of eigenvalues of the introduced boundary-value problems that are less than $\alpha - \varepsilon$, the relations $v(\varepsilon) < k(\varepsilon)$, $k'(\varepsilon) < v'(\varepsilon)$, which, in combination with (7), give the double inequality

$$v(\varepsilon) < m(\varepsilon) < v'(\varepsilon). \tag{8}$$

For each of the cylinders composing the domain Λ_ε, the determination of the number of eigenvalues less than $\alpha - \varepsilon$ reduces to the solution, with respect to the natural m_i, of the inequality

$$\frac{\alpha}{[1+\delta(z_i)]^2} + \frac{m_i^2 \pi^2}{(z_i - z_{i-1})^2} < \alpha - \varepsilon \qquad (i = 1, 2, \dots, n).$$

Arguing analogously in the case of the domain Λ'_ε, we easily derive from (8) the double inequality

$$-n + \frac{1}{\pi}\check{S}_n(\varepsilon) < m(\varepsilon) < \frac{1}{\pi}\hat{S}_n(\varepsilon) + n, \tag{9}$$

where $\check{S}_n(\varepsilon)$ and $\hat{S}_n(\varepsilon)$ are the lower and upper Darboux sums for the integral on the right-hand side of relation (11) (see Theorem 2 below).

Relation (9), under certain additional assumptions concerning the regularity of the decrease of the perturbation $\delta(z)$ at infinity, leads to an asymptotic formula for $m(\varepsilon)$. In order to facilitate the formulation of these conditions, let us denote by Γ the class of all infinitesimals of the form

$$\frac{1}{z^{k_0}} = \frac{1}{\ln_0^{k_0} z}, \quad \frac{1}{\ln_1^{k_1} z}, \quad \frac{1}{\ln_2^{k_2} z}, \dots,$$

where $z \to \infty$, $0 < k_0 < 2$, $\ln_j z = \ln(\ln_{j-1} z)$, $0 < k_j < \infty$ $(j = 1, 2, \dots)$. A function $f(z)$, defined for $z > 0$, will be called subordinate to the scale Γ if in the class Γ there is a function $\dfrac{1}{\ln_i^{k_i} z}$ such that, for any $\beta > 0$,

$$\lim_{z\to\infty} f(z) \ln_i^{k_i - \beta} z = 0, \qquad \lim_{z\to\infty} f(z) \ln_i^{k_i+\beta} z = \infty. \tag{10}$$

Theorem 2. *If, for large z, the function $\delta(z)$ in (1) is positive, decreases monotonically, and is subordinate to the scale Γ, then for the number $m(\varepsilon)$ of eigenvalues of the operator $-\hat{\Delta}$ preceding the point $\alpha - \varepsilon$, as $\varepsilon \downarrow 0$ the asymptotic formula holds*

$$m(\varepsilon) = \frac{1 + o(1)}{\pi} \int_0^{a(\varepsilon)} \sqrt{\alpha - \varepsilon - \frac{1}{[1+\delta(z)]^2}}\, dz. \tag{11}$$

Obviously, the interval of integration on the right-hand side of (11) coincides with the region of positivity of the radicand under the integral. Theorem 2 remains valid under a certain weakening of the condition of subordination to the scale Γ, when it is assumed that relations (10) hold only for some $\beta < \beta_0(i, k_i)$.

Formula (11) is analogous to the asymptotic formula recently obtained by N. Rosenfeld ([5]) in a related one-dimensional problem for a differential operator of the second order. However, in contrast to the corresponding result of N. Rosenfeld, which assumes the perturbation to be convex, twice continuously differentiable, and requires the fulfillment of a certain relation between its derivatives at infinity, Theorem 2 does not assume convexity and imposes no restrictions on the growth of the derivatives of the function $\delta(z)$, which may even fail to be differentiable. Let us also note that the above method of proof of Theorem 2, unlike the method of the paper ([5]),

based on the Prüfer transformation ([6]), makes it possible to generalize N. Rosenfeld' s asymptotic formula to two-term one-dimensional differential operators of any even order. A generalization of N. Rosenfeld' s asymptotic formula to the stationary Schrödinger equation in the whole space $\mathcal{E}$, under the assumption of fivefold differentiability of the potential and other restrictions, is given in ([7]).

3. If in (1) the factor $1 + \delta(z)$ is replaced by $\delta(z)$, then the domain Ω will contract without bound as $z \to \infty$. The spectrum of the operator $-\hat{\Delta}$ in such a domain will be discrete ([3]), and for the number $m(\lambda)$ of eigenvalues less than λ one has

Theorem 3. *If Ω is an asymptotically cylindrical domain with zero radius at infinity and finite volume V, then, under the assumption of monotone decrease of the perturbation $\delta(z)$ and uniform boundedness of $\delta'(z)$ on the half-axis $z > 0$, as $\lambda \to \infty$ the usual asymptotic formula remains valid*

$$m(\lambda) = \frac{1 + o(1)}{6\pi^2} \lambda^{3/2} V.$$

The question of the nature of the asymptotics of $m(\lambda)$ for $V = \infty$ remains unclear.

In conclusion we dwell on the question of the stability of the properties f and af with respect to finite perturbations of the boundary. It follows from Theorem 1 that when condition (4) or (5) is satisfied the indicated properties are stable, but it is of interest to consider this question in a general setting.

Let Ω be an arbitrary infinite domain of the space $\mathcal{E}$ with boundary S, and let L be the self-adjoint operator generated in $\mathcal{L}_2(\Omega)$ by the Schrödinger operation $-\Delta u + qu$ with the boundary condition $\partial u / \partial n + \rho u = 0$ or $u = 0$ on S. The part of the spectrum $S(L)$ lying to the left of the point $\lambda = 0$ is either finite (property f) or infinite (property af). If the negative part of the spectrum extends to $-\infty$ or contains at least one point of the continuous spectrum, then the invariance of property af with respect to a finite perturbation of the potential, of a finite part of the boundary, and of the boundary conditions on it follows from the splitting principle ([2]).

In the general case the question of the preservation of the properties f and af remains open. But if one assumes that, together with L, the operator generated by the operation $-(1 - \vartheta)\Delta u + qu$ (respectively, $-(1 + \vartheta)\Delta u + qu$) for some $\vartheta > 0$ and the same conditions on the boundary S of the domain Ω also has property f (respectively, af) (**the strengthened property f**, respectively, af), then the following holds.

Theorem 4. *The strengthened properties f and af of the operator L are invariant with respect to finite perturbations of the potential q, of a finite part σ of the boundary S, and of the function ρ in the boundary condition* $\partial u/\partial n + \rho u = 0$ *on σ.*

In the case when the condition $u = 0$ is prescribed on the variable part σ of the boundary S, Theorem 4 remains valid if, after the deformation, the condition $u = 0$ is preserved on σ.

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Note: Figure translations are in progress. See original paper for figures.