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Relations for the Derivatives at Points of Maximum Modulus of an Entire Transcendental Function of Many Complex Variables

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In this note we give a number of theorems of Wiman–Valiron type ([2,3]), the derivation of which is based on an idea quite different from the Wiman–Valiron method. At the same time, the central index, which plays an important role in the latter method, is replaced by another function, whose definition we shall give below.

1. Let $f(z_1, z_2, \dots, z_n)$ be an entire transcendental function in the n-dimensional complex space C_n: $z_j = x_j + iy_j$; $j = 1, 2, \dots, n$. Consider a family $V(R)$ of circular domains exhausting the space C_n, bounded by hypersurfaces $S(R)$, possessing the following two properties: 1) together with a point $(z_1, z_2, \dots, z_n)$, the domain $V(R)$ contains any point $(z_1 e^{i\vartheta_1}, z_2 e^{i\vartheta_2}, \dots, z_{ne}^{i\vartheta_n})$ with arbitrary real $\vartheta_1, \vartheta_2, \dots, \vartheta_n$, and 2) a point $(z_1, z_2, \dots, z_n) \in V(R)$ if and only if $(z_1/R, z_2/R, \dots, z_n/R) \in V(1)$.

We introduce the notation:

$$M(R) = \max_{S(R)} |f(z_1, z_2, \dots, z_n)|, \qquad K(R) = \frac{RM'(R)}{M(R)}.$$

Theorem 1. The function $\ln M(R)$ is a convex function of $\ln R$.

Theorem 2. Let $(\zeta_1, \zeta_2, \dots, \zeta_n)$ be a point at which the function $|f(z_1, z_2, \dots, z_n)|$ attains its maximum on the hypersurface $S(R)$; then

$$\sum_{j=1}^{n} \frac{\zeta_j f'_{z_j}(\zeta_1, \zeta_2, \dots, \zeta_n)}{f(\zeta_1, \zeta_2, \dots, \zeta_n)} = K(R).$$

Theorem 3. Let, as before, $(\zeta_1, \zeta_2, \dots, \zeta_n)$ be a point of maximum of the function $|f(z_1, z_2, \dots, z_n)|$ on $S(R)$, and suppose that R is a point outside some set of intervals E of finite logarithmic measure on the R-axis, the number of which on each bounded segment is finite. Then, for sufficiently large R, the function $f(z_1, z_2, \dots, z_n)$ does not vanish in the polycylinder

$$|z_j - \zeta_j| < c|\zeta_j| / K^{1+\beta}(R), \qquad j = 1, 2, \dots, n,$$

where $\beta > 0$ is an arbitrary number, and $c = c(\beta) > 0$ is independent of R.

For the formulation of the next theorem we introduce the operator $D_j = z_j \partial/\partial z_j$. The operator D_j, obviously, is additive, homogeneous, and commutative in the class of analytic functions.

Theorem 4. Outside the set E indicated in Theorem 3, the inequality

$$\left| D_1^{j_1} D_2^{j_2} \cdots D_n^{j_n} \ln f(\zeta_1, \zeta_2, \dots, \zeta_n) \right| < C\, [K(R)]^{j_1+j_2+\cdots+j_n-1+\gamma};$$

holds; here the constant $C = C(j_1, j_2, \dots, j_n)$, while $\gamma > 0$ is an arbitrarily small fixed number, $j_1 + j_2 + \cdots + j_n > 1$.

Theorem 5. Outside the set E the equalities

$$\lim_{R\to\infty} \left(\frac{\xi_1^{i_1} \xi_2^{i_2} \cdots \xi_n^{i_n} \dfrac{\partial^{i_1+i_2+\cdots+i_n}}{\partial z_1^{i_1} \partial z_2^{i_2} \cdots \partial z_n^{i_n}} f(\xi_1, \xi_2, \dots, \xi_n)}{[K(R)]^{i_1+i_2+\cdots+i_n} f(\xi_1, \xi_2, \dots, \xi_n)} - \prod_{i=1}^{n} \alpha_i^{i_i} \right) = 0,$$

where $\alpha_j = \alpha_j(R) \ge 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$.

2. Let $V_{a_1 a_2 \dots a_n}(R)$ be an exhaustive space, and let C_n be a family of "almost circular" domains possessing property 1) and the modified property 2): a point $(z_1, z_2, \dots, z_n) \in V_{a_1 a_2 \dots a_n}(R)$ if and only if $(z_1/R^{a_1}, z_2/R^{a_2}, \dots, z_n/R^{a_n}) \in V_{a_1 a_2 \dots a_n}(1)$. By $S_{a_1 a_2 \dots a_n}(R)$ we denote the hypersurface bounding the domain $V_{a_1 a_2 \dots a_n}(R)$. In proving the following theorems the results of § 1 are used. We denote:

$$M_{a_1 a_2 \dots a_n}(R) = \max_{S_{a_1 a_2 \dots a_n}(R)} |f(z_1, z_2, \dots, z_n)|, \qquad K_{a_1 a_2 \dots a_n}(R) = \frac{R M'_{a_1 a_2 \dots a_n}(R)}{M_{a_1 a_2 \dots a_n}(R)}.$$

Theorem 6. $\ln M_{a_1 a_2 \dots a_n}(R)$ is a convex function of $\ln R$.

Theorem 7. Let $(\xi_1, \xi_2, \dots, \xi_n)$ be a point at which the maximum of $f(z_1, z_2, \dots, z_n)$ is attained on the hypersurface $S_{a_1 a_2 \dots a_n}(R)$. Then

$$K_{a_1 a_2 \dots a_n}(R) = \sum_{j=1}^{n} a_j \frac{\xi_j f'_{z_j}(\xi_1, \xi_2, \dots, \xi_n)}{f(\xi_1, \xi_2, \dots, \xi_n)}.$$

Further, a theorem analogous to Theorem 4 holds, and the following one:

Theorem 8. With the exception of a certain set of intervals of the R-axis of finite logarithmic measure, the number of which on every bounded segment is finite, the relations hold:

$$\lim_{R\to\infty}\left(\frac{\xi_1^{i_1}\xi_2^{i_2}\cdots\xi_n^{i_n}\dfrac{\partial^{i_1+i_2+\cdots+i_n}}{\partial z_1^{i_1}\partial z_2^{i_2}\cdots\partial z_n^{i_n}}f(\xi_1,\xi_2,\dots,\xi_n)}{[\overline{K}(R)]^{i_1+i_2+\cdots+i_n}f(\xi_1,\xi_2,\dots,\xi_n)}-\prod_{j=1}^{n}\alpha_j^{i_j}\right)=0,$$

where $\alpha_j=\alpha_j(R)\geq 0$; $a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n=1$; $\overline{K}(R)=K_{a_1a_2\dots a_n}(R)$.

3. The theorem formulated below contains an exceptional set E in the space $(r_1,r_2,\dots,r_n)$, possessing the following properties:

a) the set E is a set of domains whose intersection with any straight line passing through the origin consists of a set of intervals of finite logarithmic measure (the number of intervals on every bounded segment of this line is finite); b)

$$\int\frac{dQ}{t^n}<\infty,$$

where dQ is the element of volume of the space $E(r_1,r_2,\dots,r_n)$, $t=(r_1^2+r_2^2+\cdots+r_n^2)^{1/2}$; c) in any bounded closed domain $\overline{D}$ of the space $(r_1,r_2,\dots,r_n)$ the number of domains of the set E belonging to $\overline{D}$ is finite.

Let

$$M(r_1,r_2,\dots,r_n)=\max|f(z_1,z_2,\dots,z_n)|;\qquad |z_j|=r_j;\quad j=$$

$$=1,2,\dots,n\quad\text{and}\quad K(r_1,r_2,\dots,r_n)=\sum_{j=1}^{n}\frac{r_j\,\partial M/\partial r_j}{M}.$$

Below, the manner in which the point $(r_1,r_2,\dots,r_n)$ tends to infinity is restricted by the following condition: $K(r_1,r_2,\dots,r_n)\to\infty$ as $r_1+r_2+\cdots$

$\dots+r_n\to\infty$, for the fulfillment of which it is necessary that the modulus of at least one of the variables with respect to which the function $f(z_1,z_2,\dots,z_n)$ is transcendental tend to infinity. This will also be sufficient if $r_1r_2\cdots r_n>0$.

Under these conditions theorems analogous to Theorems 3 and 4 hold. Denote by $(\xi_1,\xi_2,\dots,\xi_n)$ the point at which
$\max|f(z_1,z_2,\dots,z_n)|$ is attained for $|z_j|=r_j$; $j=1,2,\dots,n$.

Theorem 9.

$$K(r_1,r_2,\dots,r_n)=\sum_{j=1}^{n}\frac{\xi_jf'_{z_j}(\xi_1,\xi_2,\dots,\xi_n)}{f(\xi_1,\xi_2,\dots,\xi_n)}.$$

Theorem 10. *Outside the exceptional set E the relations hold*

$$\lim_{r_1+r_2+\dots+r_n\to\infty}\left(\frac{\xi_1^{i_1}\xi_2^{i_2}\cdots\xi_n^{i_n}\frac{\partial^{i_1+i_2+\dots+i_n}}{\partial z_1^{i_1}\partial z_2^{i_2}\cdots\partial z_n^{i_n}}f(\xi_1,\xi_2,\dots,\xi_n)}{[K(r_1,r_2,\dots,r_n)]^{i_1+i_2+\dots+i_n}f(\xi_1,\xi_2,\dots,\xi_n)}-\prod_{j=1}^{n}\alpha_j^{i_j}\right)=0,$$

where $\alpha_j=\alpha_j(r_1,r_2,\dots,r_n)\ge 0,\quad \alpha_1+\alpha_2+\dots+\alpha_n=1.$

Remark 1. The theorems of the last paragraph 3 and paragraphs 1, 2, generally speaking, do not overlap, since different exhaustions are used.

Remark 2. The theorems stated include also the case of one variable.

4. In the proof of all the theorems set forth, a lemma of a type known in the theory of increasing functions of Borel ([1]) is used essentially. In the formulation given below this lemma was not found by us.

Lemma. *Let $h(x)$ be a positive function, continuous from the right on the half-axis $x>0$, increasing to infinity together with x. Then, with the exception of a certain set of intervals E of finite logarithmic measure, the number of which on each bounded segment is finite, the inequality holds*

$$h\left(x+\frac{x}{h^{1+\alpha}(x)}\right)<h(x)+c,$$

where $\alpha>0,\ c>0$ are arbitrary numbers, on which E depends.

It is interesting to note that the logarithmic measure is uniformly bounded with respect to the functions $h(x)$, i.e., the logarithmic measure of the exceptional set is bounded above by a number which does not depend on the function $h(x)$.

5. Generalizations. We give below some generalizations of the theorems stated for functions of one variable. Analogous generalizations are also possible for functions of many variables.

1) Our theorems are true if the function $f(z)$, analytic outside some circle $|z|\le R_0$, has as singular points only branch points, and the modulus $|f(z)|$ is a single-valued function. In every bounded domain of the z-plane lying outside the circle $|z|\le R_0$, the function $f(z)$ is bounded and the number of branch points is finite. Here we require that

$$\lim_{r\to\infty}K(z)=\infty.$$

Thus, for example, our theorems are applicable to the function

$$f(z)=\prod_{j=1}^{\infty}\left(1-\frac{z}{a_j}\right)^{\alpha_j},$$

where $\alpha_j \geq \alpha > 0,\ j = 1, 2, \dots$, are arbitrary positive numbers and

$$\sum_{j=1}^{\infty} \frac{\alpha_j}{|a_j|} < \infty.$$

2) The theorems are valid for functions analytic in the angle $\varphi_1 < \varphi < \varphi_2$, where

$$\max_{|z|=r} |f(z)|$$

is attained at points $\zeta = re^{i\varphi(r)}$ satisfying the condition: $\varphi_1 + \delta < \varphi(r) < \varphi_2 - \delta$, where $\delta > 0$ is an arbitrarily small number. In this case we require: $\lim_{r\to\infty} K(r) = \infty$.

3) The theorems are valid for functions analytic in a strip H (possibly curvilinear) of width h, under the assumption that $\max |f(z)|$ on arcs of the circles $|z| = r$ in H is attained at points whose distance from the boundary of the strip is not less than an arbitrary positive number $\delta > 0$.

In addition, we require that

$$\lim_{r\to\infty} \frac{r}{K^{1+\alpha}(r)} = 0$$

for every $\alpha > 0$.

The Wiman–Valiron method does not permit such generalizations.

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Note: Figure translations are in progress. See original paper for figures.