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## Abstract

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## MATHEMATICS

**V. I. SHEVCHENKO**

# ON A LOCAL HOMEOMORPHISM OF THREE-DIMENSIONAL SPACE REALIZED BY A SOLUTION OF A CERTAIN ELLIPTIC SYSTEM

*(Presented by Academician I. N. Vekua on 10 VII 1962)*

In the present paper we consider a certain analogue of the complex Beltrami equation in three-dimensional space

$$DU - Q\bar{D}U = 0, \tag{1}$$

where $U(x)$ is an unknown four-component real vector, and the operators $D$ and $\bar{D}$ are formed with the aid of the matrices

$$\gamma_1 = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}, \qquad \gamma_2 = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix}, \qquad \gamma_3 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}$$

and have the form

$$D = \sum_{i=1}^{3} \gamma_i \frac{\partial}{\partial x_i}, \qquad \bar{D} = \sum_{i=1}^{3} \bar{\gamma}_i \frac{\partial}{\partial x_i},$$

(the bar above denotes transposition).

$Q(x)$ is a real matrix of order four,

$$Q(x) = \sum_{i=1}^{3} \bar{\gamma}_i q_i(x),$$

defined at each point $x = (x_1, x_2, x_3)$ of a finite domain $G$ of three-dimensional Euclidean space $E_3$. The equation $DU = 0$ was considered by A. V. Bitsadze, who was interested mainly in boundary-value problems (see ($^2$)).

Suppose that the system (1) is elliptic in the sense of Petrovsky in the closed domain $G$. Then

$$\det\left(\sum_{i=1}^{3} \Gamma_i \xi_i\right) = -(1 - r^4)(\xi_1^2 + \xi_2^2 + \xi_3^2)^2,$$

where $\Gamma_i = \gamma_i - Q\bar{\gamma}_i$, and from ellipticity there follows the requirement

$$r(x) = \sqrt{q_1^2(x) + q_2^2(x) + q_3^2(x)} \le q_0 < 1, \qquad q_0 = \text{const.} \tag{2}$$

Regarding $Q(x)$ we shall assume that the $q_i(x)$ possess generalized derivatives $\partial q_i/\partial x_k$ $(i, k = 1, 2, 3)$ in the sense of S. L. Sobolev, with $\partial q_i/\partial x_k \in L_p(\bar{G})$, $p > 3$, and that the inequality (2) is fulfilled. Hence, in particular, it follows that $Q \in C_\alpha(\bar{G})$. In what follows, by $\partial/\partial x_k$ we shall always mean a generalized derivative and shall consider, generally speaking, generalized solutions of equation (1).

Using the ideas of the book ($^1$), we shall show that equation (1) always admits a solution, any three components of which realize a local homeomorphism of the space $E_3$ onto the space defined by these components.

Consider the operator

$$T\omega = -\frac{1}{4\pi} \iint_G \bar{D} \frac{1}{|x - \xi|} \omega(\xi)\, d\xi.$$

Here $\omega$ and $T\omega$ are four-component vectors.

If $\omega \in L_p(\bar{G})$, $p > 1$, then, as B. V. Boyarskii showed, the function $\psi(x) \equiv T\omega$ has (generalized) derivatives $\partial\psi(x)/\partial x_s \in L_p(\bar{G})$, and

$$\frac{\partial \psi(x)}{\partial x_s} = -\frac{1}{4\pi} {}^{*}\!\iint_G \frac{\partial}{\partial x_s}\left(\bar{D} \frac{1}{|x - \xi|}\right) \omega(\xi)\, d\xi + \frac{1}{3}\bar{\gamma}_s \omega(x), \qquad x \in G, \tag{3}$$

where the integral is understood in the sense of the principal value, and the estimate holds ($^3$, $^4$)

$$L_p\left(\frac{\partial \psi}{\partial x_s}\right) \le B_p L_p(\omega), \tag{4}$$

where $B_p$ does not depend on $\omega$ (we use the notation of ($^1$)).

Introduce the operators

$$\Pi_s\omega \equiv \frac{\partial\psi}{\partial x_s}, \qquad s = 1, 2, 3;$$

$$\Pi\omega \equiv g(x) = \sum_{s=1}^{3} \bar{\gamma}_s \Pi_s \omega.$$

From (3), for $\omega \in L_p(\bar{G})$, there follows the representation ($x \in G$)

$$\Pi\omega = -\frac{1}{4\pi}\, {}^{*}\!\iint_G \bar{D}^2 \frac{1}{|x-\xi|}\,\omega(\xi)\, d\xi + \frac{1}{3}\sum_{s=1}^{3} \bar{\gamma}_s^2 \omega,$$

and if $\omega \in C_\alpha(\bar{G})$, then

$$\Pi\omega = -\frac{1}{4\pi} \iint_G \bar{D}^2 \frac{1}{|x-\xi|}[\omega(\xi) - \omega(x)]\, d\xi + \Phi(x)\omega(x), \tag{5}$$

where

$$\Phi(x) = \frac{1}{4\pi} \int_S \bar{D}S_\xi \cdot \bar{D} \frac{1}{|x-\xi|}.$$

Here $S$ is a Lyapunov surface bounding the domain $G$, and

$$\bar{D}S_\xi = \sum_{i=1}^{3} \bar{\gamma}_i \alpha_i \, dS_\xi;$$

$\alpha_i$ are the direction cosines of the exterior normal to $S$.

From inequality (4) it follows that

$$L_p(\Pi\omega) \le C_p L_p(\omega). \tag{6}$$

We shall need the following properties of the operators $T\omega$ and $\Pi\omega$. Let $\omega \in L_p(\bar{G})$, $p > 3$. Then ($x, y \in E_3$)

$$|\psi(x) - \psi(y)| \le M L_p(\omega)|x-y|^\alpha, \qquad \alpha = \frac{p-3}{p}. \tag{7}$$

If $\omega \in C_\alpha(\bar{G})$ and the boundary of the domain $G$ is sufficiently smooth, then

$$T\omega \in C^1_\alpha(\bar{G}); \tag{8}$$

$$|g(x) - g(y)| \leq M'_\alpha H(\omega, \alpha)|x - y|^\alpha \quad (x, y \in G), \tag{9}$$

i.e. $\Pi\omega \in C_\alpha(\bar{G})$ and

$$C_\alpha(\Pi\omega) \leq M_\alpha C_\alpha(\omega). \tag{10}$$

These relations are proved analogously to the plane case (see ($^1$)). For $\Pi_s\omega$ the same estimates hold as for $\Pi\omega$.

We shall seek a solution of equation (1) in the form

$$U = Z + T\omega, \qquad \omega \in L_2(\overline{G}), \tag{11}$$

where

$$Z = \left\| \begin{matrix} x_1 + x_2 + x_3 \\ x_2 \\ x_3 \\ x_1 \end{matrix} \right\|.$$

It is easy to compute that $DZ = 0$, $\overline{D}Z = 2\left\| \begin{matrix} 0 \\ 1 \\ 1 \\ 1 \end{matrix} \right\| = \mu$.

From (3) it follows that $DT\omega = \omega$. Substituting (11) into (1), we obtain for $\omega$ the singular integral equation

$$\omega - Q\Pi\omega = Q\mu. \tag{12}$$

We shall show that equation (12), as a linear equation in $L_2(\overline{G})$, is always solvable.

It is proved that $C_2 = 1$, i.e. the norm of the operator $\Pi\omega$ in the space $L_2(\overline{G})$ is equal to one. Then $L_2(Q\Pi\omega) \leq q_0 L_2(\Pi\omega) \leq q_0 L_2(\omega) \leq L_2(\omega)$, and, by the contraction mapping principle, equation (1) has a unique solution $U(x)$ of the form (11) in the space $L_2(\overline{G})$.

Reducing equation (1) to an integral equation with the operator $T\omega$ alone, it is not difficult to show, using (7) and $\partial q_i/\partial x_k \in L_p(\overline{G})$, $p > 3$, that the constructed solution $U(x) \in C_\beta(\overline{G})$ for some $\beta$, $0 < \beta < 1$.

Let us formulate the main theorem:

**Theorem 1.** *Let $G_0$ be a neighborhood of some fixed point $x_0$ and let $Q \in C_\alpha(\overline{G}_0)$. Suppose, moreover, that*

$$|q_1(x_0)| + |q_2(x_0)| + |q_3(x_0)| < 1. \tag{13}$$

*Then in some sufficiently small neighborhood $G_0'$ ($G_0' \subset G_0$) of the point $x_0$ there exists a solution of equation (1), any three components of which realize a local homeomorphism of the space $E_3$ into the space determined by these components. This solution belongs to the class $G_\alpha^1(\overline{G}_0^{\,\prime})$, $0 < \alpha < 1$.*

Let us outline the proof of the theorem. Without loss of generality, one may take the point $x_0$ to be the origin of coordinates. Denote $q_1(0) = a$, $q_2(0) = b$, $q_3(0) = c$, and in equation (1) perform a change of the unknown functions by setting $AU = V$, where $A = e - a\gamma_1 - b\gamma_2 - c\gamma_3$. Since $\det A = 1 - r_0^4$, $r_0^2 = a^2 + b^2 + c^2$, it follows from (2) that the matrix $A^{-1}$ exists. Equation (1) then takes the form

$$DV = Q_1 \widetilde{D} V = 0, \tag{14}$$

where $\widetilde{D}V = \overline{D}A^{-1}V$ and $Q_1(0) = 0$.

Let $G_\delta$ be the closed ball with center at the origin. Denote by $C_\alpha^0(G_\delta)$ the set of vectors $\omega \in C_\alpha(E_3)$ that vanish outside $G_\delta$ and satisfy the additional condition $\omega(0) = 0$.

A solution of equation (14) in the ball $G_\delta$ is sought in the form (11), where $\omega \in C_\alpha^0(G_\delta)$, which leads to the singular integral equation for $\omega$

$$\omega - Q_1 \widetilde{\Pi}\omega = Q_1 \nu, \tag{15}$$

where $\widetilde{\Pi}\omega = \overline{D}A^{-1}T\omega$ and $\nu = \overline{D}A^{-1}Z$.

Using (9) and (10), as well as the estimate for $Q_1(x)$ in the ball $G_\delta$, we prove that for sufficiently small fixed $\delta$ equation (15) has, moreover, a unique solution in the space $C_\alpha(G_\delta)$.

By virtue of (8), the constructed solution $V \in C_\alpha^1(G_\delta)$. Therefore equation (1) has in the ball $G_\delta$ the solution

$$U = A^{-1}(Z + T\omega), \tag{16}$$

belonging to the same class $C_\alpha^1(G_\delta)$.

Denote by $A_j^{-1}$ the matrix of size 3 × 4 which is obtained from the matrix $A^{-1}$ by deleting the $j$-th row. Then the vector

$$U_j = A_j^{-1}(Z + T\omega) \qquad (j = 1, 2, 3, 4)$$

will have three components, among which the $j$-th component of the vector $U$ is not included. The Jacobian $\Delta_j$ of each transformation $U_j(x)$ of the space $E_3$ is computed directly and estimated with the aid of inequalities (2) and (13). For sufficiently small fixed $\delta$, all four Jacobians $\Delta_j$ $(j = 1, 2, 3, 4)$ in the ball $G_\delta$ are simultaneously different from zero.

**Theorem 2.** *Let* $U(x) = (p(x), u(x), v(x), w(x))$ *be a holomorphic vector in the domain* $G$ *(see (2)) and let* $\Phi(U) \equiv \Phi(p, u, v, w)$ *be a smooth vector whose components* $\Phi^1, \Phi^2, \Phi^3, \Phi^4$ *depend on* $p, u, v, w$*. In order that the vector* $\Phi(U(x))$ *be holomorphic for every* $U(x)$*, it is necessary and sufficient that* $\Phi$ *satisfy the system of equations:*

$$\begin{aligned} &\Phi_p^1 = \Phi_u^2 = \Phi_v^3 = \Phi_w^4, \\ &\Phi_p^2 = -\Phi_u^1 = -\Phi_v^4 = \Phi_w^3, \\ &\Phi_p^3 = \Phi_u^4 = -\Phi_v^1 = -\Phi_w^2, \\ &\Phi_p^4 = -\Phi_u^3 = \Phi_v^2 = -\Phi_w^1. \end{aligned} \tag{17}$$

Conditions (17) are the monogeneity conditions for the quaternionic function $\hat{\Phi}(\hat{U}) = \Phi^1 + i\Phi^2 + j\Phi^3 + k\Phi^4$ of the quaternionic argument $\hat{U} = p + iu + jv + kw$ (see (5)). But, as shown in (6), every such function has the form $\hat{\Phi}(\hat{U}) = \hat{U}M + N$, where $M$ and $N$ are constant quaternions. Therefore, for any holomorphic $U$, only the vector

$$\Phi(U) = BU + \Phi_0, \tag{18}$$

where $\Phi_0$ is a constant vector and the matrix $B$ has the form

$$B = \begin{Vmatrix} m_1 & -m_2 & -m_3 & -m_4 \\ m_2 & m_1 & m_4 & -m_3 \\ m_3 & -m_4 & m_1 & m_2 \\ m_4 & m_3 & -m_2 & m_1 \end{Vmatrix}. \tag{19}$$

Let now $Q$ be a constant matrix in the domain $G$. Then, together with the smooth solution (11), equation (1) will have the solution

$$W(x) = BA^{-1}(Z + T\omega) + W_0,$$

where $B$ is an arbitrary matrix of the form (19) and $W_0$ is a constant vector.

In conclusion, I express my deep gratitude to Academician I. N. Vekua for his constant attention to this work.

Novosibirsk State University

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*Note: Figure translations are in progress. See original paper for figures.*

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