MATHEMATICS

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. I. VINOGRADOV

ON MERTENS' FORMULA

(Presented by Academician I. M. Vinogradov on 27 XI 1961)

In this note an improvement will be given of the remainder term in the asymptotic formula for a segment of Euler' s product.

The Mertens relation is known:

$$\prod_{p \le x}\left(1-\frac{1}{p}\right)=\frac{e^{-c}}{\ln x}\left(1+O\left(\frac{1}{\ln x}\right)\right). \tag{a}$$

Using certain analytic considerations and the recent theorems of I. M. Vinogradov on the boundary of the zeros of $\zeta(s)$, the remainder term in (a) can be replaced by a quantity of order

$$\exp\left(-a_0\sqrt[3]{\ln x}\right),$$

where a_0 is an absolute constant.

This result will be obtained as a consequence of the more general Theorem 1, a brief proof of which will be given here.

Theorem 1. If $s=\sigma+it$ with the condition

$$\sigma \ge 1-\frac{c_1}{(\ln t_1)^{2/3}}, \qquad t_1=|t|+e,$$

$$|t| \le \exp(c_2 \ln x \cdot \ln\ln x),$$

where c_1, c_2 are absolute positive constants, then the equality holds:

$$\prod_{p \le x}\left(1-\frac{1}{p^s}\right)\zeta(s)(s-1)=\frac{e^{-c}}{\ln x}e^{\omega(s)}(1+\theta(s,x)), \tag{1}$$

where c is Euler' s constant,

$$\omega(s) = \int_L \frac{x^{-(w-1)} - 1}{w-1}\, dw;$$

L is the line segment joining the points s and 1; $\theta(s, x)$ is a function continuous in s and satisfying the estimate

$$|\theta(s, x)| < \exp\left(-c_3 \frac{\ln x}{(\ln x \cdot t_1)^{2/3}}\right).$$

Moreover, if $t \neq 0$, then in the indicated region the second equality holds:

$$\zeta(s) \prod_{p \le x} \left(1 - \frac{1}{p^s}\right) = \exp\left(\int_{L_1} \frac{x^{-(w-1)}}{w-1}\, dw\right) (1 + \theta(s, x)),$$

where L_1 is the straight line $\sigma' + it$ with the condition $\sigma \le \sigma' < \infty$.

Proof. Introduce the function

$$\Pi(s, x) = \prod_{p<x} \frac{1}{\left(1 - \frac{1}{p^s}\right)}. \tag{2}$$

We compute its logarithmic derivative

$$\frac{\Pi'}{\Pi}(s, x) = -\sum_{n \le x} \frac{\Lambda(n)}{n^s} - \sum_{p \le x} \sum_{p^n > x} \frac{\ln p}{p^{ns}}. \tag{3}$$

The second sum in (3) has order of magnitude

$$x^{-(\sigma - 1/2)}. \tag{4}$$

For the first sum in (3), in our domain the following equality holds:

$$-\sum_{n \le x} \frac{\Lambda(n)}{n^s} = \frac{\zeta'}{\zeta}(s) - \frac{x^{1-s}}{1-s} + O\left(x^{1-\sigma} \exp\left(-c_3 \frac{\ln x}{(\ln x \cdot t_1)^{2/3}}\right)\right). \tag{5}$$

Substituting (4) and (5) into (3) and integrating both sides of the equality along the line
$\sigma \le \operatorname{Re} s < \infty$ for $t \neq 0$, we obtain

$$\Pi(s, x) = \zeta(s) \exp\left(-\int_L \frac{x^{-(w-1)}}{w-1}\, dw\right) \left(1 + O\left(\exp\left(-c_3' \frac{\ln x}{(\ln x \cdot t_1)^{2/3}}\right)\right)\right). \tag{6}$$

We transform

$$\int_L \frac{x^{-(w-1)}}{w-1}\,dw$$

into an integral along the straight segment L_1 joining the point $\left(\sigma = 1 + \frac{1}{\ln x},\ t = 0\right)$ and the point s. We obtain

$$\int_L = \int_{L_1} + \int_1^\infty \frac{e^{-u}}{u}\,du. \tag{7}$$

We transform the integral over the segment L_1 into an integral over the segment L_2, joining the point s with the point $(1, 0)$:

$$-\int_{L_1} = \ln \frac{1}{(s-1)\ln x} - \int_0^1 \frac{1-e^{-u}}{u}\,du + \omega(s). \tag{8}$$

Substituting (5) and (7) into equality (6), we obtain Theorem 1 for the case $t \neq 0$. But in equality (1) the functions on the right and on the left are continuous in s throughout the indicated domain, and, consequently, by continuity it remains meaningful on the segment $\operatorname{Im} s = 0$.

Letting $s \to 1$ in equality (1), we obtain:

$$\prod_{p \le x}\left(1 - \frac{1}{p}\right) = \frac{e^{-c}}{\ln x}\left(1 + O\left(\exp\left(-a_0 \sqrt[3]{\ln x}\right)\right)\right).$$

We note that if the Riemann hypothesis is true, then the remainder term has order

$$x^{-1/2+\varepsilon}.$$

Leningrad Branch
of the V. A. Steklov Mathematical Institute
of the Academy of Sciences of the USSR

Received
22 XI 1961

REFERENCES

1. I. M. Vinogradov, *DAN*, **118**, No. 4 (1958).

2. A. I. Vinogradov, *Matem. sbornik*, **41** (83), 1 (1957).

3. E. C. Titchmarsh, *The Riemann Zeta-Function*, IL, 1953.

Note: Figure translations are in progress. See original paper for figures.