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Abstract

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MATHEMATICS

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A UNIQUENESS THEOREM FOR A NORMAL FROMMER DOMAIN OF THE SECOND TYPE

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In studying the behavior of integral curves of a first-order differential equation in a neighborhood of a singular point, one often has to deal with the so-called distinction problems for Frommer normal domains ([1],[2]). In the present note we shall consider the distinction problem for a normal domain of the second type (a domain N_2). The formulation of this problem is apparent from the following Hartman–Wintner theorem ([3]).

Theorem on the existence of an O-curve in N_2. Let, in the equation

$$\alpha(r)\frac{d\varphi}{dr} = \Psi(r,\varphi) \tag{1}$$

1) the function $\Psi(r,\varphi)$ be defined and continuous in r,φ in the domain

$$0 < r \le \rho, \qquad -\delta \le \varphi \le \delta, \tag{N}$$

where ρ and δ are positive numbers; $\dfrac{1}{\Psi(r,\varphi)}$ is bounded for sufficiently small r outside any fixed neighborhood of the point $O(0,0)$;

2) $\Psi(r,-\delta) > 0,\ \Psi(r,\delta) < 0$ for $0 < r \le \rho$;

3) the function $\alpha(r)$ be defined, continuous, and positive for $0 < r \le \rho$,

$$\int_0^\rho \frac{dr}{\alpha(r)} = +\infty.$$

Then there exists at least one solution $\varphi(r)$ of equation (1), which is defined on the interval $0 < r \le \rho$, and every such solution has the property $\varphi(r) \to 0$ as $r \to 0$ (i.e., it is an O-curve of equation (1)).

Under the conditions of the Hartman-Wintner theorem, the domain (N) is, for equation (1), a Frommer normal domain of the second type (a domain N_2). In it equation (1) has: a) either a unique O-curve, b) or an infinite set of O-curves. Thus there arises the problem of distinguishing these two possible arrangements of the integral curves of equation (1) in the domain N_2. We note that if $\Psi(r, \varphi)$ in the domain (N) does not increase with increasing φ for each fixed r, then there exists a unique O-curve (Peano' s criterion). In the general case the problem remains open. Lohn ([2],[4]), R. E. Vinograd and D. M. Grobman ([5]), A. F. Andreev ([6],[7]), I. S. Kukles ([8],[9]), and D. M. Gruz ([10]) gave a series of sufficient criteria for uniqueness of the O-curve in N_2. These criteria partially overlap, but do not completely cover one another (there are errors in the papers of Kukles and Gruz; see ([11])). Below we shall prove a sufficient uniqueness criterion that includes, as special cases, the criteria of Lohn and of Vinograd-Grobman, as well as the criteria of Kukles and Gruz (corrected).

We shall consider the equation

$$\alpha(r)\frac{d\varphi}{dr} = \Phi(\varphi)(1 + \beta(r, \varphi)) + \psi(r, \varphi) \equiv \Psi(r, \varphi) \tag{2}$$

under the following assumption.

Condition A. 1) The function $\alpha(r)$ is defined, continuous, and positive for $0 < r \leq \rho_1$; ρ_1 is a constant,

$$\int_0^{\rho_1} \frac{dr}{\alpha(r)} = +\infty.$$

2) The function $\Phi(\varphi)$ is defined and continuous on the interval $[-\delta_1, \delta_1]$, $\delta_1 > 0$ a constant; $\Phi(0) = 0$; for $\varphi \neq 0$, $\varphi\Phi(\varphi) < 0$, $|\Phi(\varphi)| \geq a|\varphi|^k$, where a and k are positive numbers.

3) The function $\beta(r, \varphi)$ is defined in the region

$$0 < r \leq \rho_1, \qquad -\delta_1 \leq \varphi \leq \delta_1, \tag{3}$$

is continuous there with respect to r, φ, and satisfies the inequality $|\beta(r, \varphi)| \leq \varepsilon < 1$, where ε is a constant.

4) The function $\psi(r, \varphi)$ is defined in the region (3) and is continuous there with respect to r, φ; $\psi(r, \varphi) \to 0$ as $r \to 0$, uniformly with respect to $\varphi \in [-\delta_1, \delta_1]$.

Under these conditions, for sufficiently small ρ and δ, the region (N) will be, for equation (2), a region N_2, since in it the conditions of the Hartman-Wintner theorem are fulfilled. Consequently, equation (2) has in the region (N) at least one O-curve.

Uniqueness theorem for an O-curve in N_2. Let, for equation (2):

1) condition A be fulfilled;

2)

$$\frac{\psi(r,\varphi)}{\omega^\sigma(r)} \to 0 \quad \text{as } r \to 0 \tag{4}$$

uniformly with respect to $\varphi \in [-\delta, \delta]$, where $\omega(r)$ is a fixed function of class C^1 on $(0, \rho]$; $\omega(r) > 0$; $\omega'(r) > 0$ for $0 < r \leq \rho$; $\omega(r) \to 0$ as $r \to 0$; $\sigma > 0$ is a fixed number*;

3) in the region $0 < r \leq \rho$, $|\varphi| \leq u_0 \omega^{\sigma/k}(r)$, where $u_0 > 0$ is any small number, for $\varphi_2 > \varphi_1$

$$\Psi(r,\varphi_2) - \Psi(r,\varphi_1) \leq \frac{\sigma}{k}\Lambda(r)(\varphi_2 - \varphi_1), \tag{5}$$

where the function $\Lambda(r)$ is continuous on $(0, \rho]$ and satisfies the inequality

$$\Lambda(r) \leq \frac{\alpha(r)\omega'(r)}{\omega(r)}.$$

Then equation (2) has a unique O-curve in the region (N).

Proof. Dividing both sides of equation (2) by $\omega^\sigma(r)$ and taking into account that, for $|\varphi| \geq u_0 \omega^{\sigma/k}(r)$, where $u_0 > 0$ is any fixed number,

$$\frac{|\Phi(\varphi)|}{\omega^\sigma(r)} \geq a \frac{|\varphi|^k}{\omega^\sigma(r)} \geq a u_0^k,$$

we find that in the region

$$0 < r \leq \rho, \qquad |\varphi| \geq u_0 \omega^{\sigma/k}(r) \tag{6}$$

for any integral curve $\varphi = \varphi(r)$ of equation (2), $\varphi(r)\varphi'(r) < 0$, if ρ is sufficiently small. Consequently, from the region (6) integral curves cannot approach the point $O(0,0)$.

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* From the assumptions concerning $\psi(r,\varphi)$ it follows that such functions $\omega(r)$ satisfying condition (4) exist.

To study equation (2) in the domain

$$0 < r \leqslant \rho, \qquad |\varphi| \leqslant u_0 \omega^{\sigma/k}(r),$$

we transform it by the substitution

$$\varphi = u\omega^{\sigma/k}(r).$$

In doing so we obtain the equation

$$\frac{\omega(r)}{\omega'(r)} = u' = U(r,u) \equiv \frac{\Phi\left(u\omega^{\sigma/k}(r)\right)\left(1+\beta\left(r, u\omega^{\sigma/k}(r)\right)\right) + \psi\left(r, u\omega^{\sigma/k}(r)\right)}{\alpha(r)\omega^{\sigma/k-1}(r)\omega'(r)} - \frac{\sigma}{k}u, \tag{7}$$

which must be studied in the domain

$$0 < r \leqslant \rho, \qquad |u| \leqslant u_0 \tag{8}$$

(where $u_0 > 0$ is an arbitrarily small number); namely, it is necessary to show that in this domain it has a unique solution $u(r)$, defined for all $r \in (0, \rho]$.

But for equation (7) in the domain (8) all the conditions of the Hartman-Wintner theorem are fulfilled. In particular, for any number ε_0 ($0 < \varepsilon_0 < u_0$) one can indicate a number $\rho_0 > 0$ such that in each of the two domains $0 < r \leqslant \rho_0$, $\varepsilon_0 \leqslant |u| \leqslant u_0$ both terms on the right-hand side of equation (7) have the same sign, opposite to the sign of u. Consequently, in these domains $|U(r,u)| \geqslant \frac{\sigma}{k}|u| \geqslant \frac{\sigma}{k}\varepsilon_0$, i.e. $\frac{1}{U(r,u)}$ is bounded. According to the Hartman-Wintner theorem, any solution of equation (7) which lies in the domain (8) and is defined for all $r \in (0, \rho]$ has the property $u(r) \to 0$ as $r \to 0$. But equation (7) has in the domain (8) a unique solution possessing this property, since for it in this domain the conditions of Peano' s criterion are fulfilled: for $u_2 > u_1$

$$U(r,u_2) - U(r,u_1) \leqslant \left[\frac{\sigma}{k}\Lambda(r) - \frac{\omega(r)}{\alpha(r)\omega'(r)} - \frac{\sigma}{k}\right](u_2 - u_1) \leqslant 0.$$

The theorem is proved.

In the formulation of the theorem there occur the functions $\omega(r)$ and $\Lambda(r)$. The first characterizes the smallness of the function $\psi(r,\varphi)$ as $r \to 0$, the second the smallness of the Lipschitz coefficient of the right-hand side. As follows from the theorem, these two functions are organically connected with each other. Choosing various functions $\omega(r)$, we find that equation (2) will have in the domain (N) a unique O-curve for the following functions $\Lambda(r)$:

$\omega(r)$	$o(1)$	$\frac{1}{\ln\lvert\ln r\rvert}$	$\frac{1}{\lvert\ln r\rvert}$	r	$e^{-1/r}$
$\Lambda(r)$	$\frac{\alpha(r)\omega'(r)}{\omega(r)}$	$\frac{\alpha(r)}{r\lvert\ln r\rvert\ln\lvert\ln r\rvert}$	$\frac{\alpha(r)}{r\lvert\ln r\rvert}$	$\frac{\alpha(r)}{r}$	$\frac{\alpha(r)}{r^2}$

From the table it is seen, in particular, that the smallness as $r \to 0$ of the Lipschitz coefficient of the right-hand side is a necessary condition for the uniqueness of the O-curve of equation (2) in the domain N_2 only when the order of smallness of the function $\psi(r,\varphi)$ is not high. If the function $\psi(r,\varphi)$ has a sufficiently high order of smallness as $r \to 0$, then uniqueness is preserved even with an unbounded Lipschitz coefficient.

In works ($^{2-9}$) the problem of uniqueness of the O-curve in N_2 was considered, as a rule, for the equation

$$r\frac{d\varphi}{dr} = -a\varphi^k(1+\varepsilon(\varphi)) + \psi(r,\varphi), \tag{9}$$

where $k \geqslant 1$ is an odd number; a is a positive number; $\varepsilon(\varphi)$ is an analytic function of φ in a neighborhood of the point $\varphi = 0$; $\varepsilon(0) = 0$; $\psi(r,\varphi)$ satisfies condition A. This equation is a special case of equation (2). Its right-hand side will satisfy condition (5) if the function $\psi(r,\varphi)$ satisfies this condition.

Applying our uniqueness theorem to equation (9), we obtain from it, for $\omega(r) = O(1)$, the Lonn lemma $^{(2,4)}$; for $\omega(r) = r$, the Vinograd-Grobman theorem $^{(5)}$; for

$$\omega(r) = \frac{1}{|\ln r|}$$

and for

$$\omega(r) = \frac{1}{\ln\frac{1}{r}\,\ln\ln\frac{1}{r}\,\ldots\,\underbrace{\ln\ln\cdots\ln}_{n}\frac{1}{r}}$$

an amended version for two theorems of Kukles $^{(8,9)}$ (see also $^{(11)}$).

Applying our theorem to the equation

$$r\,\frac{d\varphi}{dr} = -a\varphi^k(1+\beta(r,\varphi)) + \psi(r,\varphi),$$

where the number $k \geqslant 1$ is odd; the number $a > 0$; $\beta(r,\varphi)$ and $\psi(r,\varphi)$ satisfy condition A, we obtain from it, for $\omega(r) = r$, an amended version of one of Gruz' s theorems $^{(10)}$ (see also $^{(11)}$).

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Note: Figure translations are in progress. See original paper for figures.