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Abstract

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REGULARIZATION OF PRODUCTS OF GENERALIZED FUNCTIONS OF CAUSAL TYPE

(Presented by Academician N. N. Bogolyubov, 30 XI 1961)

The problem considered below arose in quantum electrodynamics. Bearing in mind applications specifically to field theory, the author has partly retained both the physical terminology and the formulation of the problem.

By a generalized function of causal type we shall understand a function of the form

$$g(x) = \int_0^\infty h(t) e^{-ix^2/4t}\, dt, \tag{1}$$

where $x^2 = \sum_{j=1}^n \varepsilon_j x^{j2}$, $\varepsilon_j = 1$ for $j \le r$ and $\varepsilon_j = -1$ for $j > r$. With respect to $h(t)$ it is assumed that $\int_A^\infty h(t)\, dt < \infty$ for any $A > 0$, and for small t

$$h(t) = t^{-\lambda} \varphi(t),$$

where $\varphi(t)$ is bounded, and $\lambda < n/2 + 1$. Under these assumptions $g(x)$ will represent a continuous functional on the Schwartz space S ([1]), if, by definition, one sets

$$(g, \varphi) = \int_0^\infty h(t)\, dt \int e^{-ix^2/4t} \varphi(x)\, dx.$$

The function $g(x)$, for $x^2 = x^{02} - x^{12} - x^{22} - x^{32}$ and $h(t) = -(4\pi t)^{-2} e^{-im^2 t}$, is the causal function of a scalar field.

In the present work the problem of regularizing a product of the form

$$\prod_{l=1}^N g_l \left(\sum_{k=1}^m \alpha_k^l x_k \right), \tag{2}$$

is considered, where g_l are functions of type (1) or their derivatives, α_k^l are real numbers not simultaneously vanishing for each fixed l, and x_k are vectors of an n-dimensional space.

To solve the problem posed, we shall apply a method consisting of the following. Each of the functions g_l in (2) is replaced by an average of the form

$$\bar{g}(x) = \int g(x-\xi)\psi(\xi)\,d\xi, \tag{3}$$

where $\psi(\xi) \in S$, although this condition is not in fact obligatory. For the product of averages of the form (2) obtained in this way, the possibility of passing to the limit on some subspace of the space S is studied, as $\psi(\xi) \to \delta(\xi)$. But in such a general formulation the problem is ill-posed. It is necessary to narrow the class of admissible functions $\psi(\xi)$.

If, as $\psi(\xi)$ in (3), one chooses a function with Fourier transform of the form

$$\overline{\psi}(k) = \exp\left(-\sum_{j=1}^{n} z_j k^{j2}\right),$$

where $\operatorname{Re} z_j > 0$, then for $\bar{g}(x)$ it is not difficult to obtain—

compute the representation

$$\bar{g}(x) = \int_0^\infty h(t)t^{n/2} \exp\left\{\frac{1}{4}\sum_{j=1}^{n} \frac{\varepsilon_j x^{j2}}{it - \varepsilon_j z_j}\right\} \frac{dt}{\prod_{j=1}^{n}(t + i\varepsilon_j z_j)^{1/2}}. \tag{4}$$

The function $\bar{g}(x)$ is infinitely differentiable. This property is also preserved for purely imaginary z_j, if $\operatorname{Im}(\varepsilon_j z_j) < 0$.

Of particular interest in what follows are means of two kinds, namely:

$$\bar{g}(x) = \int_0^\infty f(s)\,ds \int_0^\infty h(t)t^{n/2} \exp\left\{-\frac{i}{4}\sum_{j=1}^{n} \frac{\varepsilon_j x^{j2}}{t + i\varepsilon_j s^j}\right\} \frac{dt}{\prod_{j=1}^{n}(t + i\varepsilon_j s^j)^{1/2}}, \tag{5}$$

$$\bar{g}(x) = \int_0^\infty f(s)\,ds \int_0^\infty h(t)t^{n/2} \exp\left\{-\frac{i}{4}\sum_{j=1}^{n} \frac{\varepsilon_j x^{j2}}{t + s^j}\right\} \frac{dt}{\prod_{j=1}^{n}(t + s^j)^{1/2}}. \tag{6}$$

The function $f(s)$ in (5) and (6) is chosen so that $g(x)$ have several continuous derivatives. Any sequence of functions $f(s)$ tending to $\delta(s)$ on the space of continuous functions bounded for $s \geq 0$ gives rise to a sequence of means $\bar{g}(x) \to g(x)$.

There is a close connection between means of the two kinds. If $f(s)$ is analytic in the right half-plane and has proper behavior at infinity, then the integral over the positive half-axis in (5) may be shifted to the corresponding imaginary half-axes. Thus, starting from (5), one may arrive at a mean of type (6).

Thus, let us consider the functional generated by the product (2) after replacing all g_l by means (5) (or (6)) or by the corresponding derivatives of them. This functional, when applied to a certain function $\varphi(x) \in S$, is represented as a sum of integrals of the form

$$\int_0^\infty \prod_{l=1}^N f_l(s_l) h_l(t_l) t_l^{n_l/2} \frac{ds_l\, dt_l}{\prod_{j=1}^n (u_l^j)^{\lambda_{jl}}} \int P(x) \exp\left\{ -\frac{i}{4} \sum_{j=1}^n \varepsilon_j A_j(x) \right\} \varphi(x)\, dx. \tag{7}$$

Here $P(x)$ is a certain polynomial arising in differentiating the means, $\lambda_{jl} > 0$ are certain half-integers, and $A_j(x)$ is a quadratic form of the form

$$A_j(x) = \sum_{l=1}^N \frac{\left(\sum_{k=1}^m \alpha_k^l x_k^j \right)^2}{u_l^j}. \tag{8}$$

For means (5), $u_l^j = t_l + i\varepsilon_j s_l^j$, while for means (6), $u_l^j = t_l + s_l^j$.

In what follows, concerning the forms $\sum_{k=1}^m \alpha_k^l x_k^j$, we shall assume that among them there are m linearly independent ones. Otherwise, choosing the independent forms as new variables and integrating in (7) with respect to the corresponding x_k, we arrive at the required situation.

By Parseval' s equality,

$$\int \exp\left\{ -\frac{i}{4} \sum_{j=1}^n A_j(x) \varepsilon_j \right\} P(x) \varphi(x)\, dx =$$

$$= (2\sqrt{\pi})^n e^{i\frac{\pi}{4}(n-2r)} \frac{1}{\prod_{j=1}^n \sqrt{\operatorname{Det} A_j}} \int \exp\left\{ i \sum_{j=1}^n \varepsilon_j A_j^{-1}(p) \right\} P\left(-i\frac{\partial}{\partial p} \right) \widetilde{\varphi}(p)\, dp. \tag{9}$$

Here $A_j^{-1}(p)$ is the quadratic form inverse to $A_j(x)$, and since $|A_j(x)|$ is bounded below on the unit sphere by a positive number for any finite $s_l^i \geq 0$ and $t_l > 0$, $A_j^{-1}(p)$ is bounded above at each such point. The coefficients of $A_j^{-1}(p)$, consequently, are also bounded and are homogeneous functions of degree one in u_l^i.

If $P(x)$ in (9) is simply $\prod x_k^j$ (among the factors there may also be identical ones), then, integrating by parts on the right the required number of times, we arrive at the expression

$$\sum_m \prod_{j=1}^{n} \frac{R_j^m(u^j)}{\sqrt{\operatorname{Det} A_j}} \int \exp\left\{ i \sum_{j=1}^{n} \varepsilon_j A_j^{-1}(P) \right\} \prod_k^{(m)} p_k^j \, \widetilde{\varphi}(p)\, dp.$$

With respect to $R_j^m(u^j)$ it is not difficult to prove that these are homogeneous functions of positive degree of homogeneity.

Moreover, the following assertion is valid:

Whatever the product (2) may be, one can always choose $\prod x_k^j$ so that the functions

$$\frac{R_j^m(u^j)}{\prod\limits_{l=1}^{N} (u_l^j)^{\lambda_{jl}} \sqrt{\operatorname{Det} A_j}}$$

will be uniformly bounded in every finite part of the domain $t_l > 0$, $s_l^i \geq 0$.

Hence the following follows immediately.

Theorem. Every product of the form (2), composed of N functions of type (1) and their derivatives, generates a linear continuous functional on some subspace S_N of the form $\prod x_k^j \cdot S$, where the choice of $\prod x_k^j$ is determined by the product itself.

It is not difficult to see that the resulting functional on S_N does not depend on the choice of the sequences of functions $f(s)$ that generate the means of the form (5) (or (6)). It can be shown that the covariant regularization procedure as well (see ([2,3])) gives the same functional on S_N. But in this case the limiting functional is defined on a domain broader than S_N.

If in (3) one takes

$$\psi(\xi) = \frac{1}{(\pi s)^{n/2}} \exp\left[-\frac{\sum\limits_{1}^{n} \xi_i^2}{4s} \right],$$

then $\bar{g}(x) \to g(x)$ as $s \to \infty$. For such means one can show that the domain of definition of the limiting functional corresponding to (2) will be the same as in the case of the covariant procedure.

The method proposed above also makes it possible to obtain results on noncovariant regularization of the S-matrix set forth in the papers ([4]).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.