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## Abstract

## Full Text

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*MATHEMATICS*

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# INVESTIGATION OF THREE-DIMENSIONAL BOUNDARY-VALUE PROBLEMS POSSESS- ING AXIAL SYMMETRY

*(Presented by Academician I. N. Vekua on 12 IV 1962)*

1. We investigate the elliptic system of equations

$$\begin{aligned} \frac{\partial}{\partial x}(rV_x) + \frac{\partial}{\partial r}(rV_r) &= rg_1(x, r), \\ \frac{\partial}{\partial x}V_r - \frac{\partial}{\partial r}V_x &= g_2(x, r) \end{aligned} \quad (1)$$

in a certain domain  $G$ , situated in the upper half-plane  $r \geq 0$ . This system describes a three-dimensional vector field possessing axial symmetry, the right-hand side of system (1) characterizing a prescribed distribution of sources and vortices in the meridian half-plane  $r \geq 0$ ,  $r^2 = y^2 + z^2$ . On the part  $\Gamma$  of the boundary  $\partial G$  of the domain  $G$  that does not lie on the axis  $r = 0$ , a general linear condition is prescribed,

$$\alpha V_x + \beta V_r = \gamma, \quad (2)$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$  are prescribed functions on  $\Gamma$ .

Let  $\Gamma$  consist of a finite number of curves having no common points and no points of self-intersection. Some of them  $\Gamma_1, \dots, \Gamma_m$  lie strictly above the axis  $r = 0$  and may contain both closed and open\* piecewise-smooth arcs. The other part  $\Gamma^{(1)}, \dots, \Gamma^{(n)}$  consists of a finite number of simple unclosed arcs whose ends rest on the axis  $r = 0$ . We denote by  $\Gamma_0$  the collection of segments of the axis  $r = 0$  entering into the boundary of the domain  $G$ , so that  $\partial G = \Gamma + \Gamma_0$ . The order of connectivity of the domain  $G$  is equal to  $(m + 1)$ . The domain  $G$  itself may be either finite or infinite. Under sufficiently broad assumptions on  $g_1$ ,  $g_2$ , problem (1), (2) can be reduced to the case in which system (1) is homogeneous. In complex notation we obtain the equation

$$\frac{\partial f}{\partial \bar{z}} - \frac{1}{4ir}f - \frac{1}{4ir}\bar{f} = 0, \quad f = V_r + iV_x, \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial r} \right) \quad (3)$$

and the boundary condition

$$\operatorname{Re}[a(z)f(x, r)] = \gamma(z), \quad z \in \Gamma, \quad a = \beta - i\alpha. \quad (4)$$

**2.** Let  $\bar{\Gamma}$  denote the contour symmetric with respect to the axis  $r = 0$  to the contour  $\Gamma$ . We extend  $a(z)$  to  $\bar{\Gamma} + \Gamma$ , setting  $a(z) = \overline{a(\bar{z})}$ ,  $z \in \Gamma$ . We shall study problem (3), (4) in two cases:  $\alpha_1$ )  $a(z)$  is continuous in the sense of Hölder at every point of  $\Gamma + \bar{\Gamma}$ ;  $\alpha_2$ )  $a(z) = a_1(z) dz/ds$ , where  $a_1$  satisfies condition  $\alpha_1$ , and  $z = z(s)$  is the equation of the contour referred to arc length.

**Lemma 1.** *Suppose that  $\Gamma$  consists of piecewise-smooth (in the sense of Hölder) curves, and that  $\Gamma + \bar{\Gamma}$  has discontinuous curvature at the points of intersection with the axis  $r = 0$ . Then every continuous solution  $f(x, r)$  of the homogeneous problem (3), (4) ( $\gamma(z) \equiv 0$ ) has in the closed domain  $G + \partial G$  a finite number of zero points.*

\* In the latter case they are regarded as consisting of two banks.

Lemma 1 is equivalent to the assertion that the zeros of  $f$  are isolated in  $G + \partial G$ . If such a point  $z_0$  lies above the axis  $r = 0$ , then the assertion is known ((<sup>1</sup>), Ch. 4, § 4). If it lies on the axis  $r = 0$  and is interior with respect to  $\Gamma_0$ , then the assertion follows from the analyticity of  $f(x, r)$  at the point  $(x_0, 0)$  and the uniqueness theorem for equation (3). Finally, if a zero lies on the axis  $r = 0$  and is angular (between  $\Gamma$  and  $\Gamma_0$ ), then its isolation is proved by means of the method of paper (<sup>3</sup>).

**Lemma 2.** *Let  $f(x, r)$  be the same field as in Lemma 1, and let  $z_0$  be its zero. If  $z_0$  is interior with respect to  $G$ , then  $f(x, r) = (z - z_0)^n f_0$ ; if  $z_0$  is a boundary point, with  $\operatorname{Im} z_0 > 0$ , then  $f(x, r) = (z - z_0)^{n'/\nu} f'_0(x, r)$  (in case  $a_1$ ) and  $f(x, r) = (z - z_0)^{(n''+1)/\nu-1} f''_0(x, r)$  (in case  $a_2$ ). Here  $n, n', n''$  are integers  $\geq 0$ ;  $\nu\pi$  is the angle measure at the point  $z_0$  interior with respect to  $G$ ;  $f_0, f'_0, f''_0$  are continuous functions, not equal to zero at the point  $z = z_0$ .*

The first two assertions of Lemma 2 are known ((<sup>1</sup>), Ch. 4, § 4); the third is proved analogously.

**Lemma 3.** *Let  $f(x, r)$  be the same field as in Lemma 1, and let  $x_0$  be its zero, located inside  $\Gamma_0$ . Then the (integer) number*

$$\frac{1}{2\pi} [\arg f(x, r)]_{K_\rho(x_0)} = n, \quad (5)$$

where  $K_\rho(x_0)$  is the circle with center at the point  $x_0$  and radius  $\rho$ , containing no other singular points of the field  $f$ , is positive and is determined by the conditions:

$$d^k f(x_0, 0) = 0, \quad k = 1, 2, \dots, n-1; \quad d^n f(x_0, 0) \neq 0.$$

In other words, the index of the zero  $(x_0, 0)$  of the field  $f(x, r)$  is positive and is equal to the multiplicity of the zero of the function  $f(x, r)$ .

**Lemma 4.** *Let  $f(x, r)$  be a field satisfying the conditions of Lemma 1, and let  $(x_0, 0)$  be its zero, located at the intersection of  $\Gamma$  with the axis  $r = 0$ . Then in both cases  $a_1$  and  $a_2$  the representation holds*

$$f(x, r) = f_0(x, r) + o\left[\left((x - x_0)^2 + r^2\right)^{n/2}\right], \quad (6)$$

where  $n$  is a positive integer,  $f_0(x, r)$  is a solution of equation (3) in a full neighborhood of  $(x_0, 0)$ , and  $d^n f_0(x_0, 0) \neq 0$ .

3. Let  $N_G$  be the sum of all integers  $n$  at the interior zeros of the field  $f(x, r)$ , satisfying conditions (3), (4) ( $\gamma \equiv 0$ );  $N_\Gamma$  the sum of all integers  $n$  corresponding, according to Lemma 2, to the boundary zeros of the field  $f(x, r)$  lying above the axis  $r = 0$ ;  $N_0$  the sum of all integers  $n$  entering formula (5) and corresponding to zeros of the field  $f$  interior with respect to  $\Gamma_0$ ;  $N_\infty$  the multiplicity of the zero of the field  $f$  at infinity;  $N$  the sum of all integers  $n$  entering formula (6) and corresponding to zeros of the field  $f(x, r)$  located at the angles on the axis  $r = 0$ . In terms of these quantities we formulate the assertion:

**Theorem 1.** *Let  $f(x, r)$  be a solution, continuous in  $G + \partial G$ , of the homogeneous problem (3), (4) ( $\gamma(z) \equiv 0$ ). Then:*

- 1) In case  $a_1$ :

$$4N_G + 2N_\Gamma + 2N_0 + N = 2\mu \quad \text{for } N_\infty = 0,$$

$$4N_G + 2N_\Gamma + 2N_0 + 2N_\infty + N = 2\mu + 2 \quad \text{for } N_\infty > 0, \quad (7)$$

where

$$\mu = \frac{1}{\pi} \sum_{i=1}^m [\arg a(z)]_{\Gamma_i} + \frac{1}{\pi} \sum_{i=1}^n [\arg \overline{a(z)}]_{l_i}. \quad (8)$$

- 2) In case  $a_2$ :

$$4N_G + 2N_\Gamma + 2N_0 + N = 4m + 2n + 2\kappa \quad \text{for } N_\infty = 0,$$

$$4N_G + 2N_\Gamma + 2N_0 + 2N_\infty + N = 4m + 2n + 2\kappa + 2 \quad \text{for } N_\infty > 0, \quad (9)$$

where

$$\kappa = \frac{1}{\pi} \sum_{i=1}^m [\arg a_1(z)]_{\Gamma_i} + \frac{1}{\pi} \sum_{i=1}^n [\arg a_1(z)]_{\Gamma_0^{(i)}}. \quad (10)$$

In the proof of Theorem 1, Lemmas 1-4 are used in an essential way, and it is assumed that the domain  $G$  is unbounded. The theorem admits a generalization to the case where the field  $f(x, r)$  has in  $G + \partial G$  a finite number of singularities of pole type. The case of a finite domain  $G$  is also considered. Let us additionally note that, in computing the quantities (8), (10), the curves  $\Gamma_i, \Gamma_0^{(i)}$  are traversed in the positive direction relative to the domain  $G$ .

Among the large number of consequences of Theorem 1, let us give, for example, the following: if  $\kappa < 0$  (in case  $\alpha_1$ ) or  $2m + n + \kappa < 0$  (in case  $\alpha_2$ ), then the homogeneous problem (3), (4) (for  $\gamma \equiv 0$ ) has no nontrivial solutions in the class of functions continuous in  $G + \partial G$  and vanishing at infinity. The assertion follows directly from formulas (7), (9), if one notes that in the case under consideration  $N_\infty \geq 2$ .

4. Passing to the inhomogeneous problem (3), (4), we restrict ourselves, for simplicity, to the case where  $n = 0$ ,  $m \geq 1$ , and  $\Gamma$  has no corner points. We shall seek the solution in the class of functions: a) regular at every point; b) continuous up to the axis  $r = 0$  and having at infinity a uniformly attainable limit  $iV_x^\infty$ ; c) continuous up to the contour  $\Gamma$ .

An important role is played by the adjoint problem: to determine a solution of the equation

$$\frac{\partial f'}{\partial z} + \frac{1}{4ir} f' - \frac{1}{4ir} \overline{f'} = 0, \quad (11)$$

adjoint to equation (3), satisfying on  $\Gamma$  the condition

$$\operatorname{Re} \left[ \overline{a(z)} \frac{dz}{ds} f'(x, r) \right] = 0. \quad (12)$$

Thus, if problem (3), (4) is considered in case  $\alpha_1$ , then the adjoint problem belongs to case  $\alpha_2$ , and conversely. The adjoint class of functions is determined in the case under consideration by the requirements a), b), c) and, in addition, by the condition  $f'(x, 0) \equiv 0$ ,  $-\infty \leq x \leq +\infty$ .

**Theorem 2.** If problem (3), (4) has a solution, then

$$\int_{\Gamma} \overline{a(z)} \gamma(z) f^{(j)}(x, r) dz - V_x^\infty \int_{\Gamma} \operatorname{Re} (ia(z)) \overline{a(z)} \overline{f^{(j)}(x, r)} dz = 0, \quad (13)$$

where  $f^{(j)}(x, r)$ ,  $j = 1, 2, \dots, l'$ , is a complete system of solutions of problem (11), (12), linearly independent over the field of real numbers, from the adjoint class. Conversely, if for some constant  $V_x^\infty$  conditions (13) are satisfied, then there exists a solution of problem (3), (4) from the class a), b), c), taking at infinity the value  $iV_x^\infty$ .

Denote by  $l$  the maximal number of linearly independent (over the same field) solutions of the homogeneous problem (3), (4) in the class of functions vanishing at infinity.

**Theorem 3.** The relation

$$l - l' = 2\kappa - m, \quad (14)$$

holds, where

$$\kappa = \frac{1}{2\pi} \sum_{i=1}^m [\arg a(z)]_{\Gamma_i}. \quad (15)$$

Let us note additionally that analogous theorems have also been proved without the above-indicated restrictions on the smoothness of the contour  $\Gamma$ . The method of investigation is based on reducing the problem to singular integral equations extended over the contour  $\Gamma$ . The principal role in this reduction is played by the generalized Cauchy formula for equation (3), established in paper (2). The actual determination of solutions of the problem is reduced to solving the indicated equations.

5. Comparing Theorems 1, 2, and 3 yields numerous consequences, of which we note the following:

I. If the number (15) is nonpositive, then  $l = 0$ ,  $l' = m - 2\kappa$ ; problem (3), (4) can have no more than one solution.

II. If the number (15) is greater than or equal to  $m$ , then  $l' = 0$ ,  $l = 2\kappa - m$ ; the nonhomogeneous problem is always solvable, and the homogeneous one has  $l = 2\kappa - m$  linearly independent solutions.

III. For all values of the number (15) we have the inequality

$$\max(0, 2\kappa - m) \leq l \leq 2\kappa.$$

In particular, the problem on the flow past bodies of revolution by an axisymmetric flow of an incompressible fluid leads to the (in general, nonhomogeneous) problem (3), (4) with  $a(z) = dz/ds$ . In this case we have the following assertion:

IV. The homogeneous problem on the flow past bodies of revolution by an axisymmetric flow in the class of functions bounded at infinity has exactly

$m + 1$  linearly independent solutions. The nonhomogeneous problem (corresponding, for example, to a vortical flow outside the body) is solvable for any right-hand side  $\gamma(z)$ .

This last problem, as is well known, reduces to the exterior Neumann problem. However, as consequence IV shows, the solution of the Neumann problem must be sought in the class of multivalued functions; otherwise we may lose  $m$  independent solutions. From this point of view, instead of the potential it is more convenient to consider the velocity vector field itself, since it is always single-valued. The case when the contour  $\Gamma$  contains unclosed arcs is also considered.

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*Note: Figure translations are in progress. See original paper for figures.*

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