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Abstract

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MATHEMATICS

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THE METHOD OF GRIDS FOR SOLVING PARABOLIC-TYPE EQUATIONS OF ORDER $2m$ WITH SEPARABLE VARIABLES

(Presented by Academician S. L. Sobolev on 27 X 1961)

In solving boundary-value problems for equations of parabolic type in the case of spatial variables ($p \geq 2$), the usual explicit and implicit difference schemes require a large amount of computational work, since for explicit schemes there are strong restrictions on the time step, while for implicit schemes at each layer one has to solve systems of linear equations with a number of unknowns $\sim 1/h^p$. In recent years, for the heat-conduction equation in the region $Q_T = \overline{\Omega} \times [0, T]$ ($\overline{\Omega}$ is a parallelepiped), absolutely stable difference schemes have been obtained ($^{1-4}$), for which the amount of computational work in passing from one layer to another is $\sim 1/h^p$.

In the present paper a difference scheme is proposed which is a generalization of the scheme (^{2}) to the case of a general parabolic equation with separable variables.

Let it be required, in the region $Q_T = \overline{\Omega} \times [0, T]$, $\overline{\Omega}$ $(0 \leq x_s \leq 1,\ s = 1, 2, \dots, p)$, to find a solution of the equation

$$\frac{\partial u(x,t)}{\partial t} = \sum_{s=1}^{p} L_s u(x,t) + f(x,t), \tag{1}$$

satisfying the boundary conditions

$$\left(u,\ \frac{\partial u}{\partial \nu},\ \dots,\ \frac{\partial^{m-1} u}{\partial \nu^{m-1}} \right)_S = (0, 0, \dots, 0) \tag{2}$$

and the initial condition

$$u\big|_{t=0} = \varphi(x). \tag{3}$$

Here $x = (x_1, x_2, \dots, x_p)$;

$$L_s u = \sum_{\alpha=0}^{m} (-1)^{\alpha-1} \frac{\partial^\alpha}{\partial x_s^\alpha} \left(a_{s\alpha}(x_s) \frac{\partial^\alpha u}{\partial x_s^\alpha} \right), \qquad a_{sm}(x_s) > 0;$$

$a_{s\alpha}(x_s)$ $(\alpha < m)$ are such that $L_s < 0$; S is the boundary of Q_T; ν is the normal to S.

Definition. A function $u(x,t)$ is called a generalized solution of problem (1), (2), (3) if: 1) $u \in W_2^{(1)}(Q_T)$; 2) $u \in W_2^{(m)}(\Omega_t)$ as a function of x for each $t \in [0,T]$; 3)

$$\left(u,\ \frac{\partial u}{\partial \nu},\ \dots,\ \frac{\partial^{m-1} u}{\partial \nu^{m-1}} \right)_S = (0, 0, \dots, 0)$$

in the sense of the metric $L_2(S)$; 4)

$$\lim_{\Delta t \to 0} \int_\Omega \left(u(x, \Delta t) - \varphi(x)\right)^2 d\Omega = 0;$$

5) for every function $\Phi(x,t)$ satisfying 1), 2), 3), the relation

$$\int_{Q_T} \frac{\partial u}{\partial t} \Phi \, dQ = - \int_{Q_T} \left(\sum_{s=1}^{p} \sum_{\alpha=0}^{m} a_{s\alpha} \frac{\partial^\alpha u}{\partial x_s^\alpha} \frac{\partial^\alpha \Phi}{\partial x_s^\alpha} \right) dQ + \int_{Q_T} f \Phi \, dQ$$

holds.

The notation here is the same as in (^{5,6}).

The existence of a somewhat differently defined generalized solution for a general parabolic equation and the convergence to it of approximations obtained by means of an implicit difference scheme were established in (^{7,8}). Convergence to a generalized solution can also be obtained for an explicit difference scheme, but in this case the condition $\tau/h^{2m} < c$ is required.

We now consider the following difference approximation of problem (1), (2), (3). Let τ be the time step, $h = 1/N$ the step in x_s $(s = 1, 2, \dots, p)$. Denote by $\overline{\Omega}_h$ the set of grid points $(x_s)_{i_s} = i_s h$ $(i_s = 0, 1, \dots, N)$; denote by S_h the set of grid points at which at least one index i_s is equal to $0, 1, \dots, m-1$ or $N-m+1, N-m+2, \dots, N$.

$$\Omega_h = \overline{\Omega}_h \setminus S_h; \qquad v(i_1 h, i_2 h, \dots, i_{ph}, n\tau) = v^{(n)}_{i_1 \dots i_p} = v^{(n)}_\Delta.$$

Put

$$v^{(n)}_\Delta = 0 \quad \text{for } \Delta \in S_h, \quad 0 \le n \le \frac{T}{\tau}, \tag{2'}$$

and, if $\Delta \in \Omega_h$, we shall find the function $v^{(n+1)}$ from the known function $v^{(n)}$ from the following p systems, using $p-1$ intermediate functions $v^{(n+1/p)}$:

$$\frac{v_\Delta^{(n+1/p)} - v_\Delta^{(n)}}{\tau} = L_1^h v_\Delta^{(n+1/p)} + \sum_{s=2}^{p} L_s^h v_\Delta^{(n)} + f_\Delta^{(n)},$$

$$\frac{v_\Delta^{(n+s/p)} - v_\Delta^{(n+(s-1)/p)}}{\tau} = L_s^h v_\Delta^{(n+s/p)} - L_s^h v_\Delta^{(n)}, \qquad s = 2, 3, \dots, p, \tag{4}$$

where

$$L_s^h v_\Delta = \sum_{\alpha=0}^{m} (-1)^{\alpha-1} \Delta_{-x_s}^{\alpha} \left(a_{s\alpha}(i_s h)\, \Delta_{x_s}^{\alpha} v_\Delta \right),$$

$$\Delta_{x_s} v_\Delta = \frac{v_{i_1 \dots (i_s+1) \dots i_p} - v_{i_1 \dots i_p}}{h}, \qquad \Delta_{-x_s} v_\Delta = \frac{v_{i_1 \dots i_p} - v_{i_1 \dots (i_s-1) \dots i_p}}{h}, \tag{5}$$

$$v_\Delta^{(n+s/p)} = 0 \quad \text{for } \Delta \in S_h.$$

Naturally,

$$v_\Delta^{(0)} = \varphi_\Delta. \tag{3'}$$

It is not difficult to verify that the transition by this scheme from $v^{(n)}$ to $v^{(n+1)}$ is performed at a cost of $\sim 1/h^p$ arithmetic operations. Scheme (4) is equivalent to the following scheme without the intermediate functions $v^{(n+s/p)}$:

$$\prod_{s=1}^{p} (E - \tau L_s^h) v_\Delta^{(n+1)} = \left\{ \prod_{s=1}^{p} (E - \tau L_s^h) + \tau \sum_{s=1}^{p} L_s^h \right\} v_\Delta^{(n)} + \tau_\Delta^{(n)} f_\Delta^{(n)}, \tag{6}$$

where E is the identity operator.

Under conditions (2′), $(-L_s^h)$ is a self-adjoint and positive-definite operator, and $L_{s_1}^h L_{s_2}^h = L_{s_2}^h L_{s_1}^h$ (9,10).

Considering the expansions of $v^{(n)}$ in the eigenfunctions of the operators L_s^h, one can obtain a theorem on stability with respect to the initial data.

Theorem 1. If the function $e_\Delta^{(n)}$ satisfies (6) with $f_\Delta^{(n)} = 0$, then for all k $(0 \le k \le T/\tau)$ one has $\|e^{(k)}\|_{L_2}^h \le \|e^{(0)}\|_{L_2}^h$, where $\| \ \|_{L_2}^h$ is a difference analogue of the norm in L_2.

Using the methods of (6,7), the following theorem is established:

Theorem 2. If: 1) the coefficients and the right-hand side of equation (1) are bounded in Q_T; 2) φ has in $\overline{\Omega}$ bounded partial derivatives containing no more than m differentiations with respect to each x_s; 3) φ, together with its derivatives up to order $(m-1)$, vanishes on the boundary of Ω, then for the sequence of functions $v_\Delta^{(n)}$ the inequalities

$$\left\|v_\Delta^{(n)}\right\|_{W_2^{(1),h}(Q_T)} \leq C(T) = \text{const}; \qquad \left\|v_\Delta^{(n)}\right\|_{W_2^{(m),h}(\Omega_t)} \leq C(T) = \text{const},$$

hold, where $\|v\|_{W_2^k,h(D)}$ is the difference analogue of $\|v\|_{W_2^k(D)}$.

Denote, as in (6), by $(v_\Delta)'$ the multilinear function in $x_1, x_2, \dots, x_p, t$ which coincides at the grid points with $v_\Delta^{(n)}$.

Theorem 3. Suppose: 1) the coefficients $a_{s\alpha}$ $(\alpha \leq m)$ are continuous in Q_T; 2) the functions φ, f have, respectively in Ω_0, Ω_t $(0 \leq t \leq T)$, bounded partial derivatives containing no more than $2m$ differentiations with respect to each x_s; 3) the functions φ, f, together with all their derivatives up to order $(2m-1)$, vanish respectively on the boundary of Ω_0, Ω_t. Then, if $\tau \to 0$ and $h \to 0$, the sequence

$$\left(\Delta_{x_1}^{\alpha_1} \Delta_{x_2}^{\alpha_2} \cdots \Delta_{x_p}^{\alpha_p} v_\Delta\right)'$$

for $\alpha_1 + \alpha_2 + \dots + \alpha_p \leq m-1$ tends to

$$\frac{\partial^{\alpha_1+\alpha_2+\dots+\alpha_p} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_p^{\alpha_p}}$$

in the norm $L_2(\Omega_t)$; the sequences $(\Delta_t v_\Delta)'$,

$$\left(\Delta_{x_1}^{\alpha_1} \Delta_{x_2}^{\alpha_2} \cdots \Delta_{x_p}^{\alpha_p} v_\Delta\right)' \quad (\alpha_1 + \alpha_2 + \dots + \alpha_p = m)$$

converge weakly, respectively in Q_T and Ω_t, to du/dt and to

$$\frac{\partial^{\alpha_1+\alpha_2+\dots+\alpha_p} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_p^{\alpha_p}},$$

where u is the generalized solution of (1), (2), (3).

In the proof of this theorem, along with the methods of (6), expansions in the eigenfunctions of the operators L_s^h are used. If in equation (1) $m = 1$, then Theorems 2 and 3 can be strengthened. Namely, instead of condition (2) we take the condition

$$u\big|_S = \psi(x), \tag{7}$$

and in scheme (4), for $\Delta \in S_h$, we put $v_\Delta^{(n)} = \psi_\Delta$, $v^{(n+s/p)} = \psi_\Delta$ $(s = 1, 2, \ldots, p)$. Then, with the aid of the results of (11), one establishes

Theorem 4. If, for every $t \in [0, T]$, the solution of equation (1) $(m = 1)$, satisfying conditions (3), (7), has in Q_T bounded partial derivatives containing no more than two differentiations with respect to each variable x_s $(s = 1, 2, \ldots, p)$, and the derivatives $\partial^2 u/\partial t^2$, $\partial^4 u/\partial x_s^4$, $\partial^2 a_s/\partial x_s^2$ are bounded, then, as τ and h tend to zero, the sequence $v^{(k)}$ tends in $L_2(\Omega_t)$ to $u^{(k)}$ $(k = t/\tau)$ with order of convergence $O(\tau) + O(h)$.

We note that Theorem 4 is also valid for equation (1) $(m = 1)$ with coefficients $a_{s\alpha}(x_s, t)$; if the coefficients of equation (1) are constant, then the order of convergence will be $O(\tau) + O(h^2)$.

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REFERENCES

1. D. W. Peaceman, H. H. Rachford, J. Soc. Ind. Appl. Math., 3, No. 1, 28 (1955).
2. J. Douglas, H. H. Rachford, Trans. Am. Math. Soc., 82, 421 (1956).
3. N. N. Yanenko, DAN, 125, No. 6, 1207 (1959).
4. G. A. Baker, T. A. Oliphant, Quart. Appl. Math., 17, No. 4 (1960).
5. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
6. O. A. Ladyzhenskaya, *A Mixed Problem for a Hyperbolic Equation*, M.–L., 1953.
7. O. A. Ladyzhenskaya, Mat. sborn., 39, issue 4, 491 (1956).
8. O. A. Ladyzhenskaya, Mat. sborn., 45, issue 2, 123 (1958).
9. E. G. Dyakonov, DAN, 138, No. 2, 271 (1961).
10. E. G. Dyakonov, DAN, 138, No. 3, 522 (1961).
11. V. S. Ryabenkii, A. F. Filippov, *On the Stability of Difference Equations*, 1956.

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