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Abstract

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ASYMPTOTICS OF THE SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE DERIVATIVE

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Let us consider the system

$$\mu\, dy/dt = F(y, x, t), \qquad dx/dt = f(y, x, t) \tag{1}$$

with the additional conditions

$$\alpha y(0) + \beta y(1) = y^0, \qquad x(0) = x^0. \tag{1'}$$

Here α is the column

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

in which there are n_1 ones and n_2 zeros; β is the column

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

in which there are n_2 ones and n_1 zeros, with $n_1 + n_2 = n$; y and x are vectors of dimensions n and m, respectively; μ is a small parameter.

Putting $\mu = 0$, we obtain the system

$$0 = F(y, x, t), \qquad dx/dt = f(y, x, t) \tag{2}$$

with the additional condition $x(0) = x^0$.

The solution $\{\bar{y}(t), \bar{x}(t)\}$ of system (2) is assumed known. A system similar to system (1) was considered in paper ([2]) with initial data on a certain stable manifold. In Theorem 1 of that paper the existence and uniqueness of the solution of system (1) with boundary conditions (1′) were proved; moreover, this proof was of an auxiliary character, in order subsequently to pass from conditions of the form (1′) to initial conditions, and the inequalities

$$|y(t,\mu) - \bar{y}(t)| \le \left\{K|a|\mu + |a|e^{-\sigma t/2\mu} + |b|e^{-\sigma(1-t)/2\mu} + |d|\mu\right\}, \tag{3}$$

$$|x(t,\mu) - \bar{x}(t)| \le K\{|a|\mu + |b|\mu\},$$

were also established, where $K, |a|, |b|, |d|, \sigma$ are certain positive constants.

In paper ([3]) the asymptotic behavior of a stable initial manifold was studied.

Our aim is to construct asymptotic formulas giving an approximation, uniform with respect to t, varying on the interval $0 \le t \le 1$, to the solution of system (1) of order $C\mu^{n+1}$. In obtaining the asymptotic formulas we shall also use ([1,4,5]). The assumptions under which we consider system (1) are the same as in papers ([2,3]), but with an additional requirement on the smoothness of the right-hand sides of system (1).

Thus, suppose that the conditions A are satisfied:

1) In the domain $|y - \bar{y}(t)| < \gamma$, $|x - \bar{x}(t)| < \varepsilon$, $0 \le t \le 1$, where $\gamma > 0$ and $\varepsilon > 0$ are fixed, independent of μ, but sufficiently small pos-

constant, the functions $F(y, x, t)$ and $f(y, x, t)$ have, respectively, $(n+1)$-st and n-th continuous derivatives with respect to all arguments, and on the sets

$$\begin{aligned} &|y - \bar{y}(0)| < \gamma, \qquad x = \bar{x}(0), \qquad t = 0; \\ &|\bar{y} - \bar{y}(1)| < \gamma, \qquad x = \bar{x}(1), \qquad t = 1, \end{aligned}$$

they have, respectively, $(2n+1)$-st and $2n$-th derivatives with respect to y, where

$$|\alpha(y^0 - \bar{y}(0))| < \gamma, \qquad |\beta(y^0 - \bar{y}(1))| < \gamma.$$

2) There exists a real nonsingular matrix $P(t) \in C^1$ $(0 \le t \le 1)$ such that

$$P^{-1}(t)F_y(t)P(t) = \begin{pmatrix} B(t) & 0 \\ 0 & C(t) \end{pmatrix} \qquad (0 \le t \le 1),$$

where $B(t)$ is a matrix of order $n_1 \times n_1$, $C(t)$ is a matrix of order $n_2 \times n_2$, and the characteristic roots of the matrix $B(t)$ have negative real parts, while those of the matrix $C(t)$ have positive real parts.

3) $F_y^{-1}(t)F_x(t) \in C^1$ $(0 \le t \le 1)$.

Here the following notation is used: $F = F(y, x, t)$, $\bar{F} = \bar{F}(t) = F(\bar{y}, \bar{x}, t)$, $\bar{F}_y = F_y(t)$, and analogously f, $f(t)$, $\bar{f}$, $\bar{f}_y$.

The asymptotics is constructed according to the same scheme as in [4]. Let us define the following three kinds of vector functions: a) $\bar{y}_{\mu,n}$, $\bar{x}_{\mu,n}$; b) $\overset{(n)}{y}_{\mu,k}$, $\overset{(n)}{x}_{\mu,k}$; c) y_n, x_n. We shall call the operation of formally setting μ equal to zero the degeneration of an expression.

1. First let $n = 0$.

The functions a) are defined as solutions of system (2).

The functions b) are defined as follows. We represent y and x formally in the form of series

$$\begin{aligned} y &= y_0(\tau) + y_1(\tau)\mu + \ldots + y_n(\tau)\mu^n \ldots, \\ x &= x_0(\tau) + x_1(\tau)\mu + \ldots + x_n(\tau)\mu^n \ldots. \end{aligned} \tag{4}$$

(We shall also use the notation $\overset{(n)}{y} = \mu^n y_n$, $\overset{(n)}{x} = \mu^n x_n$.) We write system (1) in the variables $\tau = \dfrac{t - t^0}{\mu}$, μ:

$$dy/d\tau = F(y, x, t^0 + \mu\tau), \qquad dx/d\tau = \mu f(y, x, t^0 + \mu\tau). \tag{5}$$

Substituting (4) into (5), expanding the right-hand sides of system (5) in μ, and collecting terms with equal powers of μ, we then obtain, for μ in the zeroth power, the system

$$dy_0/d\tau = F(y_0, x_0, t^0), \qquad dx_0/d\tau = 0,$$

or, in the variables t and μ,

$$\mu\, d\overset{(0)}{y}/dt = F(\overset{(0)}{y}, \overset{(0)}{x}, t^0), \qquad d\overset{(0)}{x}/dt = 0. \tag{6}$$

(t^0 is assigned the values 0 or 1; we agree to denote the functions corresponding to these values by the indices 0 or 1 on the upper right. We shall omit these indices when this cannot cause any misunderstanding.)

Degenerating the expression in system (6), we obtain the system

$$0 = F(\overset{(0)}{\bar{y}}, \overset{(0)}{\bar{x}}, t^0), \qquad d\overset{(0)}{\bar{x}}/dt = 0,$$

from which, under the additional condition

$$\overset{(0)}{\bar{x}}\Big|_{t=t^0} = \bar{x}(t^0),$$

the functions $\overset{(0)}{\bar{y}}$, $\overset{(0)}{\bar{x}}$ are determined; $y^{(0)}_{\mu,k}$ we define to be identically equal to zero.

The functions y_0, x_0 have not yet been determined by us, since we have not imposed additional conditions on system (5). Instead of determining them directly we find the difference $u_0 = y_0 - \bar{y}_0$ and x_0 from the system

$$du_0/d\tau = F(u_0 + \bar{y}_0, x, t^0), \qquad dx_0/d\tau = 0.$$

We take the additional conditions in the form:

$$\begin{array}{llll} \text{for } t^0 = 0 & \alpha u_0^0\big|_{\tau=0} = \alpha(y^0 - \bar{y}(0)), & \beta u_0^0\big|_{\tau=\infty} = 0, & x_0^0\big|_{\tau=0} = 0; \\ \text{for } t^0 = 1 & \beta u_0^1\big|_{\tau=0} = \beta(y^0 - \bar{y}(1)), & \alpha u_0^1\big|_{\tau=-\infty} = 0, & x_0^1\big|_{\tau=0} = \bar{x}(1). \end{array}$$

2. Suppose that we have determined functions of the form a), b), c) up to number $n-1$, and determine them for number n. Differentiating the system (1) n times with respect to μ and isolating it, we obtain

$$n\frac{d}{dt}\bar{y}_{\mu,n-1} = \bar{F}_{\mu n}, \qquad \frac{d}{dt}\bar{x}_{\mu n} = \bar{f}_{\mu n},$$

from which, under the additional condition

$$\bar{x}_{\mu n}\big|_{t=0} = (-1)^n \int_0^\infty \tau^n f^0_{n-1}\, \tau^n d\tau$$

the functions a) are determined.

Let us determine the functions b) and c). Substituting (3) into (4) and expanding the right-hand sides of system (4) in powers of μ, for μ^n we have

$$dy_n/d\tau = F_n, \qquad dx_n/d\tau = f_{n-1}, \tag{7}$$

or, in other notation:

$$\mu d\overset{(n)}{y}/dt = \overset{(n)}{F}, \qquad d\overset{(n)}{x}/dt = \overset{(n-1)}{f}.$$

Here $\overset{(n)}{F}$ and $\overset{(n-1)}{f}$ have the form:

$$\overset{(n)}{F} = \frac{1}{n!}\frac{d^n}{d\nu^n}\exp D\Big|_{\nu=0}\overset{(0)}{F},$$

$$D = d_1\nu + d_2\nu^2 + \cdots + d_n\nu^n,$$

$$d_1 = \overset{(1)}{y}\,\partial/\partial y_0 + \overset{(1)}{x}\,\partial/\partial x_0 + (t - t^0)\,\partial/\partial t_0,$$

$$d_2 = \overset{(2)}{y}\,\partial/\partial y_0 + \overset{(2)}{x}\,\partial/\partial x_0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$d_n = \overset{(n)}{y}\,\partial/\partial y_0 + \overset{(n)}{x}\,\partial/\partial x_0, \qquad \overset{(0)}{F} = F(\overset{(0)}{y}, \overset{(0)}{x}, t^0).$$

Differentiating system (7) k times with respect to μ and isolating, we obtain

$$k\frac{d}{dt}\overline{\overset{(n)}{y}_{\mu,k-1}} = \overline{\overset{(n)}{F}_{\mu k}}, \qquad \frac{d}{dt}\overline{\overset{(n)}{x}_{\mu k}} = \overline{\overset{(n-1)}{f}_{\mu k}}.$$

We take the additional conditions in the form:

$$\overline{\overset{(n)}{y}_{\mu k}}(t^0) = \begin{cases} \bar{y}_{\mu k}(t^0), & n = k, \\ 0, & n \neq k. \end{cases}$$

In what follows the notations

$$\overset{(n)}{y} = \left(\overline{\overset{(n)}{y}} + \mu \overline{\overset{(n)}{y}_{\mu}} + \cdots + \frac{\mu^n}{n!} \overline{\overset{(n)}{y}_{\mu n}}\right), \qquad \bar{y}_{nk} = \frac{\overline{\overset{(n)}{y}_{\mu}}^{\,k}}{\mu^{n-k}},$$

$$\bar{y}_n = \left(\bar{y}_{n0} + \bar{y}_{n1} + \cdots + \frac{1}{n!} \bar{y}_{nn}\right).$$

Similarly for F, x, and f.

Now let us define the functions y_n, x_n, but, just as before, instead of defining them directly we define the difference $u_n = y_n - \overset{(n)}{y}_n$ and x_n. We have the system:

$$du_n/d\tau = F_n - \overset{(n)}{F}_n, \qquad dx_n/d\tau = f_{n-1}. \tag{8}$$

The structure of the right-hand sides of system (8) is known to us; therefore system (8) can be written in the form

$$\frac{du_n}{d\tau} = F_{y_0} u_n + G_{n-1}, \qquad dx_n/d\tau = f_{n-1},$$

where G_{n-1} contains functions of index not higher than $n-1$. We take the additional conditions in the form:

$$\text{for } t_0 = 0 \qquad \alpha u_n^0\big|_{\tau=0} = -\frac{1}{n!} \overset{(n)}{y}{}^{0}_{\mu^n}, \qquad \beta u_n^0\big|_{\tau=\infty} = 0, \qquad x_n^0\big|_{\tau=0} = 0;$$

$$\text{for } t_0 = 1 \qquad \beta u_n^{1\prime}\big|_{\tau=0} = -\frac{1}{n!} \overset{(n)}{\bar{y}}{}^{1}_{\mu^n}, \qquad \beta u_n^1\big|_{\tau=-\infty} = 0,$$

$$x_n^1\big|_{\tau=0} = \frac{1}{n!}\left(\bar{x}_{\mu^n}(1) - (-1)^n \int_0^\infty \tau^n f_{n-1}^1 \tau^n \, d\tau\right).$$

Theorem. *If conditions A are satisfied, then the estimates*

$$|y - Y_n|, \; |x - X_n| < C\mu^{n+1} \qquad (0 \le t \le 1),$$

hold, where C does not depend on t and μ,

$$Y_n = \overset{(0)}{y}{}^0 + \overset{(1)}{y}{}^0 + \ldots + \overset{(n)}{y}{}^0 + \overset{(0)}{y}{}^1 + \overset{(1)}{y}{}^1 + \ldots + \overset{(n)}{y}{}^1 + \bar{y} + \mu\bar{y}_\mu + \ldots + \frac{\mu^n}{n!}\bar{y}_{\mu^n}$$
$$- \left[\overset{(0)}{\bar{y}}{}^0 + \ldots + \overset{(n)}{\bar{y}}{}^0 + \mu\left(\overset{(1)}{\bar{y}}{}^0_\mu + \ldots + \overset{(n)}{\bar{y}}{}^0_\mu\right) + \ldots + \frac{\mu^n}{n!}\overset{(n)}{\bar{y}}{}^0_{\mu^n}\right]$$
$$- \left[\overset{(0)}{\bar{y}}{}^1 + \ldots + \overset{(n)}{\bar{y}}{}^1 + \mu\left(\overset{(1)}{\bar{y}}{}^1_\mu + \ldots + \overset{(n)}{\bar{y}}{}^1_\mu\right) + \ldots + \frac{\mu^n}{n!}\overset{(n)}{\bar{y}}{}^1_{\mu^n}\right].$$

X_n has an analogous form.

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Note: Figure translations are in progress. See original paper for figures.