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Abstract

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MATHEMATICS

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TRANSFORMATIONS IN SEMIGROUPS AND MULTIVALUED MAPPINGS

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Let Ω be an arbitrary topological space and R the ring of all bounded continuous functions defined on Ω; then to each continuous mapping $f : \Omega \to \Omega$ there corresponds a homomorphism $T : R \to R$, defined as follows: $T(x)(\sigma) = X(f(\sigma))$ for all $X(\cdot) \in R$ and $\sigma \in \Omega$. Conversely, under certain conditions, to each homomorphism $T : R \to R$ there corresponds a continuous mapping $f : \Omega \to \Omega$. Thus, under these conditions the homomorphisms $T : R \to R$ represent continuous mappings $f : \Omega \to \Omega$. The aim of the present paper is to show how multivalued mappings may be represented as "generalized homomorphisms" in semigroups of a special type.

1. In what follows we shall need an additive semigroup S, in which the following conditions are satisfied:

1) $x + y = y + x;\ x, y \in S$;
2) $(x + y) + z = x + (y + z);\ x, y, z \in S$;
3) there exists a unique element $0 \in S$ such that $x + 0 = x$ for every $x \in S$.

We also assume that S is partially ordered:

4) $0 \leqslant x$ for all $x \in S$;
5) from the inequality $x \leqslant y$ follows the inequality $x + z \leqslant y + z;\ x, y, z \in S$;
6) there exists an element $e \in S$ $(e \neq 0)$ such that for any $x \in S$ there is an integer n such that

$$x \leqslant ne = \underbrace{e + \cdots + e}_{n \text{ times}};$$

7) if H is a nonempty subset of S, then $\inf H$ exists.

Remark. From condition 7) it follows that if H is a nonempty subset of S bounded above, then $\sup H$ exists.

Examples. Let Ω be an arbitrary topological space. Define the semigroup $S_1 = S_1(\Omega)$ as the collection of all bounded lower semicontinuous functions taking only nonnegative integer values. Thus, if $X(\cdot) \in S_1$ and $\sigma \in \Omega$, then $X(\sigma) \in \{0, 1, 2, ...\}$, and the set $\{\tau : x(\tau) > \alpha\}$ is open (α is an arbitrary real number).

If the order and addition of elements in S_1 are defined in the usual way, and if we choose $0(\cdot), e(\cdot) \in S_1$ so that $0(\sigma) = 0$ and $e(\sigma) = 1$ for every $\sigma \in \Omega$, then conditions 1)–7) are satisfied in S_1.

The semigroup $S_2 = S_2(\Omega)$ is defined as the collection of all bounded upper semicontinuous functions taking only nonnegative integer values. If addition of elements and the order in S_2 are defined in the usual way, then we take $0(\cdot), e(\cdot) \in S_2$ so that $0(\sigma) = 0$ and $e(\sigma) = 1$ for any $\sigma \in \Omega$. In this case conditions 1)–7) are satisfied in S_2.

Definition. A nonempty set $I \subset S$ is called an **ideal** if: 1) $x + y \in I$, when $x, y \in I$; 2) $y \in I$, when $y \leqslant x$ and $x \in I$. The ideal I is called **proper** if $I \neq S$.

Lemma 1. *If I is an ideal, then $I = S$ if and only if $e \in I$.*

It follows from this lemma that maximal proper ideals always exist. Further, if Ω is a T_1-space and $S_1 = S_1(\Omega)$, then the collection of all maximal proper ideals with the appropriate topology is the Urysohn compactification of the space Ω.

Definition. A **transformation** T in a semigroup S is any mapping T of the semigroup S into itself having the following properties: 1) $T(x+y) \leqslant T(x)+T(y)$ for all $x, y \in S$; 2) $T(nx) = nT(x)$ for every $x \in S$ and $n = 0, 1, 2, ...$; 3) $T(x) \leqslant T(y)$, if $x \leqslant y$.

We shall call a transformation $T : S \to S$ a **generalized homomorphism** if $T^{-1}(I)$ is a proper ideal in the case where I is a proper ideal. The term "homomorphism" is justified by the fact that, if T were a homomorphism of a ring R (with identity) into itself, then $T^{-1}(I)$ would be a proper ideal whenever I is a proper ideal.

It is easy to prove the following lemma:

Lemma 2. If T is a transformation in the semigroup S and $e \leqslant T(e)$, then T is a generalized homomorphism.

Definition. A transformation T in the semigroup S is called **lower semicontinuous** if, for every nonempty $H \subset S$, when H is directed upward and bounded, we have $T(\sup H) = \sup T(H)$. A transformation T is called **upper semicontinuous** if, for every nonempty $H \subset S$, we have $T(\inf H) = \inf T(H)$, when H is directed downward.

2. We shall call $g(\cdot)$ a **multivalued mapping** if to each $\tau \in \Omega$ there corresponds a nonempty closed set $g(\tau) \subset \Omega$.

Definition. A multivalued mapping $g(\cdot)$ is called **lower semicontinuous** if, for all $\sigma_0 \in \Omega$ and any open set U $(g(\sigma_0) \cap U \neq \emptyset)$, there exists an open set V

such that $\sigma_0 \in V$ and $g(\sigma) \cap U \neq \emptyset$ for all $\sigma \in V$.

Definition. A multivalued mapping $g(\cdot)$ is called **upper semicontinuous*** if, for all $\sigma_0 \in \Omega$ and any open set U $(g(\sigma_0) \subset U)$, there exists an open set V such that $\sigma_0 \in V$ and $g(\sigma) \subset U$ for all $\sigma \in V$.

Remark. If $g(\cdot)$ is a lower (or upper) semicontinuous multivalued mapping and the set $g(\sigma)$ consists of one point for every $\sigma \in \Omega$, then $g(\cdot)$ is equivalent to a single-valued continuous mapping.

Theorem 1. Let Ω be an arbitrary topological space, $S_1 = S_1(\Omega)$, and let $g(\cdot)$ be a lower semicontinuous multivalued mapping. If, for all $x(\cdot) \in S_1$, $\sigma \in \Omega$, we put

$$T(x)(\sigma) = \sup\{x(\tau) : \tau \in g(\sigma)\},$$

then T will be a lower semicontinuous transformation of S_1 into itself. T is a generalized homomorphism, since $T(e) = e$. Conversely, to every lower semicontinuous generalized homomorphism T there corresponds a lower semicontinuous multivalued mapping $g(\cdot)$.

Theorem 2. Let Ω be a bicompact Hausdorff space, $S_2 = S_2(\Omega)$, and let $g(\cdot)$ be an upper semicontinuous multivalued mapping. If, for each $x(\cdot) \in S_2$ and each $\sigma \in \Omega$, we put

$$T(x)(\sigma) = \sup\{x(\tau) : \tau \in g(\sigma)\},$$

then T will be an upper semicontinuous transformation of S_2 into itself. Since $T(e) = e$, T is a generalized homomorphism. Conversely, to every upper semicontinuous generalized homomorphism T there corresponds an upper semicontinuous multivalued mapping $g(\cdot)$.

* V. Gurevich (1) introduced these mappings under the name continuous.

3. As an application, let us consider the problem of extending multivalued mappings. Let Ω be a bicompact Hausdorff space and $S_2 = S_2(\Omega)$. Let Ω_0 be an everywhere dense subset of Ω, and let $g_0(\cdot)$ be a multivalued mapping whose domain of definition is Ω_0. It is assumed that $g_0(\cdot)$ is upper semicontinuous in the sense that, for every $\sigma_0 \in \Omega_0$ and every open set U $(g_0(\sigma_0) \subset U)$, there exists an open set V such that $\sigma_0 \in V$ and $g_0(\sigma) \subset U$ for every $\sigma \in V \cap \Omega_0$. The question arises of the possibility of extending $g_0(\cdot)$ to a multivalued mapping $g(\cdot)$ defined on the whole space Ω and upper semicontinuous. The possibility of such an extension is shown as follows: for every $x(\cdot) \in S_2$ and every $\sigma \in \Omega_0$, put

$$T_0(x)(\sigma) = \sup\{x(\tau) : \tau \in g_0(\sigma)\}.$$

Then put

$$T(x) = \inf\{y : y(\cdot) \in S_2,\ T_0(x)(\sigma) \le y(\sigma) \text{ for all } \sigma \in \Omega_0\}.$$

It can be shown that T is an upper semicontinuous generalized homomorphism, to which, according to Theorem 2, there corresponds an upper semicontinuous multivalued mapping $g(\cdot)$, and moreover $g(\sigma) = g_0(\sigma)$ for all $\sigma \in \Omega_0$.

Thus, we have obtained an extension $g(\cdot)$ of the multivalued mapping $g_0(\cdot)$. It can also be shown that $g(\cdot)$ is the minimal extension in the sense that, if $h(\cdot)$ is any extension of the mapping $g_0(\cdot)$, then $g(\sigma) \subset h(\sigma)$ for every $\sigma \in \Omega$.

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Note: Figure translations are in progress. See original paper for figures.