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Abstract

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EXCESSIVE FUNCTIONS AND ADDITIVE FUNCTIONALS OF MARKOV PROCESSES

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The concept of an excessive function associated with a Markov process was first introduced by Hunt (^{4}). In the study of excessive functions it was found that there is a one-to-one correspondence between a broad class of excessive functions and a certain subclass of the class of additive functionals (^{1,5,7}). In the present note this correspondence is subjected to further analysis.

1. Let us recall some definitions. Let $X = (x_t, \zeta, \mathcal{M}_t, P_x, \theta_t)$ be a homogeneous Markov process given in a measurable topological space $(E, \mathcal{E}, \mathcal{B})$.

We use the terminology and notation of E. B. Dynkin (^{2}). By $\mathcal{N}_t$ we denote the σ-algebra generated by the events $\{x_s(\omega) \in \Gamma,\ \zeta > s\}$, where $s \leq t,\ \Gamma \in \mathcal{B}$. The smallest σ-algebra containing all $\mathcal{N}_t$ is denoted by $\mathcal{N}^0$. By $\mathcal{N}_{t+0}$ is denoted the σ-algebra of events A such that, for every $u > t$, $\{A, \zeta > u\} \in \mathcal{N}_u$. Let $\mathfrak{N}$ be some σ-algebra ($\mathfrak{N} \subseteq \mathcal{N}^0$). We put $A \in \mathfrak{N}$ if for any finite measure μ on $\mathcal{B}$ there exist sets $A_1, A_2 \in \mathfrak{N}$ such that $A_1 \subseteq A \subseteq A_2$ and

$$\int P_x\{A_2 \setminus A_1\}\, \mu(dx) = 0.$$

By τ_U we denote the time of first hitting the set U. More precisely, $\tau_U = \zeta$ if $x_t \notin U$ for all $t > 0$, and

$$\tau_U = \inf\{t:\ t > 0,\ x_t \in U\}$$

otherwise. If $P_x\{\tau_U = 0\} = 1$, then the point x is called regular for U.

A strictly Markov, right-continuous homogeneous process is called standard (^{3}) if: a) the space $(E, \mathcal{E})$ is Hausdorff, locally compact, and has a countable base, and the σ-algebra $\mathcal{B}$ consists of its Borel sets; b) if τ_n is a nonincreasing sequence of random variables independent of the future, and $\tau = \lim_{n\to\infty} \tau_n$, then for every $x \in E$

$$x_{\tau_n} \to x_\tau$$

(P_x-almost surely on the set $\{\tau < \zeta\}$); c) $\overline{\mathcal{N}}_{t+0} \subseteq \mathcal{M}_t$.

An almost Borel-measurable function $f(x)$* $(0 \leq f(x) \leq \infty)$ is called an **excessive** function associated with X, if for all $t \geq 0$ and $x \in E$

$$f(x) \geq M_x f(x_t), \qquad f(x) = \lim_{s \to 0} M_x f(x_s). \tag{1}$$

Let E_n be a sequence of compact sets whose union is E, and such that E_n is contained in the open kernel of the compact set E_{n+1}. Denote by ζ_n the time of first hitting the set $E \setminus E_n$. An excessive function $f(x)$ is called **harmonic** if $f(x) = M_x f(x_{\zeta_n})$. This definition does not depend on the choice of $\{E_n\}$. We shall say that the harmonic function f is regular if

$$f(x) = M_x\{\lim_{t \uparrow \zeta} f(x_t)\}.$$

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* This means $(^4)$ that for any finite measure μ on $\mathcal{B}$ one can specify two Borel-measurable functions $g(x)$ and $h(x)$ such that $g \leq f \leq h$, and the probability

$$P_x\{\text{there exists } t \geq 0 \text{ for which } g(x_t) \neq h(x_t)\}$$

is equal to zero μ-almost everywhere.

We shall call a function $\varphi_t(\omega)$ $(0 \leqslant t < \zeta(\omega),\ 0 \leqslant \varphi_t(\omega) \leqslant \infty)$ a **homogeneous additive functional** (or simply a **functional**) of the process X (1), if: 1) the function φ_t is $\mathcal{N}_{t+0}$-measurable; 2) $\varphi_s + \theta_s \varphi_t = \varphi_{s+t}$, P_x-almost surely on the set $\{\omega : s + t < \zeta(\omega)\}$, for all $s, t \geqslant 0$, $x \in E$. Unless otherwise stated, we shall consider exclusively functionals φ_t that are continuous from the right, i.e., such functionals for which the function φ_t is right-continuous in t, P_x-almost surely for all $x \in E$. We shall say that the functional φ_t belongs to the class U $(^5)$, if for every finite measure μ on $\mathcal{B}$ one can indicate in the σ-algebra $\mathcal{N}^0$ two sets A_1 and A_2 such that $A_1 \subseteq \{\omega : \varphi_t$ and x_t, as functions of t, have no common discontinuity points$\} \subseteq A_2$ and

$$\int_E P_x\{A_2 \setminus A_1\} \mu(dx) = 0.$$

We shall extend the functionals φ_t to $0 \leqslant t \leqslant \infty$ by setting $\varphi_t(\omega) = \varphi_{\zeta-0}(\omega)$ for $\zeta \leqslant t \leqslant \infty$ (the quantity $\varphi_{\zeta-0}$ is defined P_x-almost surely). The function $M_x \varphi_\infty$ is excessive; it is called the potential of the functional φ_t.

An excessive function $f(x)$ will be called **singular** if, for every point x at which $f(x)$ assumes a finite value: a) there exists an increasing sequence of open sets G_n such that $P_x\{\tau_{G_n} \uparrow \zeta\} = 1$ and $M_x f(x(\tau_{G_n})) = f(x)$; b) $\lim_{t \uparrow \zeta} f(x_t) = 0$. If $f(x)$ is singular and $f(x) < \infty$, then there exists neither a regular harmonic function $g(x)$ nor a functional φ_t such that $g(x) \neq 0$, $M_x \varphi_\infty \neq 0$, and $g(x) \leqslant f(x)$, $M_x \varphi_\infty \leqslant f(x)$.

Theorem 1. Let X be a standard process. Then every excessive function $f(x)$ taking finite values* is representable in the form

$$f(x) = f_1(x) + f_2(x) + f_3(x), \tag{2}$$

where $f_1(x)$ is a regular harmonic function, $f_2(x)$ is a singular function, and $f_3(x)$ is the potential of some functional φ_t from the class U.

Remark 1. From the results of Meyer (see ([6]), Part II, § 4) it is not difficult to derive that in the decomposition (2) the functions f_1, f_2, and f_3 are chosen uniquely, and that if $f_3(x) = M_x\varphi_\infty = M_x\psi_\infty$, where $\varphi_t, \psi_t \in U$, then $P_x\{\varphi_t = \psi_t\} = 1$ for all $x \in E$ and $t \geqslant 0$.

Proof. It is readily verified that the function

$$F_1(x) = M_x\{\lim_{n\to\infty} f(x(\zeta_n))\}$$

is harmonic ([6]), Part II). If the function $f(x)$ is bounded, then the representation (2) follows from Lemma 3 of ([6]) and from Theorem 1 of ([7]); moreover $f_1(x) = F_1(x)$, and $f_3(x) = 0$. Let $f(x)$ be unbounded. The functions $h_n(x) = \min(n, f(x) - F_1(x))$ are bounded and excessive and, according to what has been said, are potentials of certain functionals $\varphi_t^{(n)} \in U$. Let G_n be some open neighborhood of the set $A_n = \{x : f(x) > n\}$, $\tau_n = \tau_{G_n}$, and let G'_n be the set of points regular for G_n. Then, for $m \leqslant n$,

$$h_n(x) = M_x \int_0^{\tau_m - 0} d\varphi_t^{(n)} + M_x \int_{\tau_m+0}^{\zeta} d\varphi_t^{(n)}, \tag{3}$$

where the first of the summands is the potential of the functional

$$\bar{\varphi}_t = \begin{cases} \varphi_t^{(n)}, & \text{for } 0 \leqslant t < \tau_m, \\ \varphi_{\tau_m - 0}^{(n)}, & \text{for } \tau_m \leqslant t < \infty, \end{cases}$$

for the part X_m of the process X on the set $E \setminus G'_n$, while the second term is a harmonic function for X_m. From Remark 1 it follows that, for $n \geqslant m$ and any $t \geqslant 0$, the functions $\varphi_t^{(n)}(\omega)$ P_x-almost surely coincide on the set $\{\omega : t < \tau_n\}$.

Using the equality $\lim_{n\to\infty} P_x\{\tau_{A_n} = \zeta\} = 1$, choose, for a fixed x, a sequence G_n such that $P_x\{\tau_n \uparrow \zeta\} = 1$, and define φ_t as $\lim_{n\to\infty} \varphi_t^{(n)}$ (P_x-almost surely). It is clear that $\varphi_t \in U$.

Put $m = n - 1$ in equality (3), and denote the second term on the right-hand side of this equality by $H_n(x)$. Letting now n tend to infinity in (3), we conclude that there exists the limit $F_2(x) = \lim_{n\to\infty} H_n(x)$, and

$$F_2(x) = f(x) - F_1(x) - M_x\varphi_\infty.$$

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* It can be shown that the condition that the function $f(x)$ be finite is not essential.

From the last relation it is seen that

$$F_2(x) = \lim_{t\to 0} F_2(x_t)$$

(P_x-almost surely). Moreover,

$$f(x) - F_1(x) \geqslant H_n(x) \geqslant \int_{t<\tau_k} H_n(x_t)\, P_x\{d\omega\},$$

if $n \geqslant k+1$. Hence it is easy to derive the inequality

$$F_2(x) \geqslant M_{xF} 2(x_t)$$

and thereby prove the excessivity of the function $F_2(x)$.

From the harmonicity of $H_n(x)$ for X_m $(m < n)$ and the inequality $f(x) - F_1(x) \geqslant H_n(x)$ it follows that $F_2(x)$ is singular.

Thus, the functions $f_1(x) = F_1(x)$, $f_2(x) = F_2(x)$, and $f_3(x) = M_x\varphi_\infty$ satisfy the condition of the theorem.

2. Consider Brownian motion Z_m in m-dimensional Euclidean space R_m $(m \geqslant 3)$. By $|x-y|$ we shall denote the Euclidean distance between the points x and y. In this case the class of excessive functions for Z_m coincides with the class of nonnegative functions superharmonic in the whole space R_m, completed by the function identically equal to $+\infty$, while the class of harmonic functions for Z_m consists of the nonnegative constants.

According to the well-known Riesz theorem ([8]), every nonnegative function $f(x)$ superharmonic in the whole R_m admits the representation

$$f(x) = C + \int_{R_m} |x-y|^{-m+2}\, \mu(dy),$$

where C is a nonnegative constant, and μ is a Borel measure; moreover, C and μ are uniquely determined by the function $f(x)$. Hence one can infer that, for $f(x)$ to be singular, it is necessary and sufficient that $C = 0$ and that the measure μ be concentrated on some set of capacity zero. If, however, $C = 0$ and the measure μ assigns zero mass to every set of capacity zero, then $f(x)$ is the potential of some functional φ_t^*.

Indeed, let μ_n be the restriction of the measure μ to the set $\{x : n < f(x) \leqslant n+1\}$. Then $\mu = \sum \mu_n$, and the functions

$$g_n(x) = \int |x-y|^{-m+2}\, \mu_n(dy)$$

are excessive and do not exceed $n+1$. From Theorem 1 there follows the existence of functionals $\varphi_t^{(n)}$ whose potentials are the functions $g_n(x)$. The expression

$$\varphi(t) = \lim_{s\to t+} \sum_{n=0}^{\infty} \varphi_s^{(n)}$$

is the required functional (cf. (6), p. 187).

From what was said in the preceding paragraph there follows the validity of the assertion of Theorem 1 also for functions $f(x)$, excessive for Z_m, which assume

* This result was first obtained by A. D. Ventcel' (9).

the value $+\infty$. In this case one should set

$$f_1(x) = C, \qquad f_2(x) = \int_A |x-y|^{-m+2}\,\mu(dy),$$

$$f_3(x) = \int_{R_m \setminus A} |x-y|^{-m+2}\,\mu(dy),$$

where C and μ have the same meaning as before, and $A = \{x : f(x) = +\infty\}$.

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REFERENCES

1. V. A. Volkonskii, *Tr. Mosk. matem. obshch.*, **9**, 143 (1960).
2. E. B. Dynkin, *Foundations of the Theory of Markov Processes*, Moscow, 1959.
3. E. B. Dynkin, *Theory of Probability and Its Applications*, **5**, No. 4, 441 (1960).
4. G. A. Hunt, *Illinois J. Math.*, **1**, No. 1, No. 3 (1957); **2**, No. 2 (1958).
5. P. A. Meyer, *C. R.*, **252**, 2279 (1960).
6. P. A. Meyer, *Ann. Inst. Fourier*, **12**, 125 (1962).
7. M. G. Shur, *DAN*, **137**, No. 4, 800 (1961).
8. M. Brelot, *Éléments de la théorie classique du potentiel*, Paris, 1959.
9. A. D. Wentzel, *DAN*, **137**, No. 1, 17 (1961).

Note: Figure translations are in progress. See original paper for figures.