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Abstract

Full Text

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ON DIRICHLET L-FUNCTIONS

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Let $\chi(n)$ be a nonprincipal character modulo D. Consider the behavior of the corresponding L-function on the line $\sigma = 1/2$ near the real axis, for example for $|t| \leq 1$.

Davenport (1) gave the estimate

$$L(1/2 + it, \chi) = O(D^{1/4}). \tag{1}$$

Yu. V. Linnik (2) somewhat improved this estimate for the case when the character $\chi(n)$ is the Kronecker symbol $\left(\frac{-D}{n}\right)$, where $-D < 0$ is a fundamental discriminant satisfying certain conditions; he gave the estimate

$$L(1/2 + it, \chi) = o(D^{1/4}). \tag{2}$$

In the present note, estimate (1) is improved for the case of L-functions for which $\chi(n)$ is a nonprincipal character modulo a prime p. We use the shortened functional equation for L-functions obtained by Yu. V. Linnik (3,4), in combination with Burgess' s method (5). The estimate of the theorem of the present paper can be replaced by an even more precise one, if the proof is made more complicated.

We introduce some notation. Let ω be any fixed number satisfying the inequalities $1 > \omega > 2/3$;

$$\alpha = \frac{1}{46+\omega}; \qquad \beta = \frac{42+\omega}{2(46+\omega)} - \delta,$$

where

$$\frac{1}{5} < \delta < \frac{18+\omega}{2(46+\omega)}; \qquad s = \frac{1}{2} + it.$$

Lemma 1. Let $\chi(n)$ be a nonprincipal character $(\mathrm{mod}\, p)$. Then, for all sufficiently large p and any N, the inequality

$$\left|\sum_{n=N+1}^{N+H} \chi(n)\right| < \frac{H}{p^{\alpha}}, \tag{3}$$

holds, where

$$p^{\beta+\delta} < H < p^{1/2+\delta}. \tag{4}$$

The proof of the lemma is similar to the proof of Theorem 1 of paper [5], with Lemma 2 of that paper replaced by Lemma 2 of paper [6].

Lemma 2. Let $|t| \leq 1$, and let $\chi(n)$ be a nonprincipal character (mod p). Then

$$L(s,\chi) = \sum_{n\leq\sqrt{p}\ln^2 p} \chi(n)n^{-s}\gamma\left(s, n\sqrt{\frac{\pi}{p}}\right) + \sum_{n\leq\sqrt{p}\ln^2 p} \overline{\chi}(n)n^{s-1}\gamma_1\left(s, n\sqrt{\frac{\pi}{p}}\right) + O\left(\frac{1}{p}\right). \tag{5}$$

The form of the functions γ_i depends only on the value $\chi(-1) = \pm 1$.

For the proof and all notation pertaining to the lemma, see (3).

Theorem. Let $|t| \leqslant 1$, and let $\chi(n)$ be a non-principal character $(\bmod\, p)$. In this case the estimate

$$L\left({}^1/_2 + it, \chi\right) = O\left(p^{\frac{42+\omega}{4(46+\omega)}}\right). \tag{6}$$

holds.

We indicate the proof of the theorem. We apply Lemma 2 and estimate only the first sum on the right in (5), since the second is estimated analogously. We split this sum into two sums

$$\sum_{1\leqslant n\leqslant p^{\beta+\delta}} \chi(n)n^{-s}\gamma\left(s, n\sqrt{\frac{\pi}{p}}\right) + \sum_{p^{\beta+\delta}<n<\sqrt{p}\ln^2 p} \chi(n)n^{-s}\gamma\left(s, n\sqrt{\frac{\pi}{p}}\right) = \Sigma_1 + \Sigma_2.$$

The sum Σ_1 is estimated trivially, and we use the inequality (3)

$$\left|\gamma\left(s, n\sqrt{\frac{\pi}{p}}\right)\right| < c, \tag{7}$$

where c is a positive constant.

In order to estimate the sum Σ_2 properly, we apply Abel’ s transformation to it, and then Lemma 1 and the following estimate (4):

$$\gamma(k) - \gamma(k+1) = O\left(\frac{\ln^3 p}{k^{3/2}}\right), \tag{8}$$

where

$$\gamma(k) = \frac{\Gamma\left(\frac{s+a}{2}\right)}{k^s}\gamma\left(s, k\sqrt{\frac{\pi}{p}}\right), \qquad a = \frac{1-\chi(-1)}{2}.$$

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Note: Figure translations are in progress. See original paper for figures.