# Reports of the Academy of Sciences of the USSR

1962

SovietRxiv

## Abstract

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# Reports of the Academy of Sciences of the USSR

1962. Volume 142, No. 3

## MATHEMATICS



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# ON THE ABSENCE OF POLYHEDRAL SPECTRA FOR BICOMPACTA

*(Presented by Academician P. S. Aleksandrov, 14 IX 1961)*

Freudenthal proved ([1]) that every compactum $\Phi$ can be represented as the limit of a simplicial, i.e. also polyhedral, spectrum $S = \{P_n, \omega_n^m\}$, $m > n$, $n = 1, 2, ...$, (for the definitions see ([2])) with projections "onto," and moreover, if $\dim \Phi \leq r$, then for every $n$ one may assume $\dim P_n \leq r$. The problem arose of extending Freudenthal' s results to arbitrary bicompacta. P. S. Aleksandrov and S. Mardešić (see ([3]), p. 240) independently posed the question whether every $r$-dimensional bicompactum is the limit of a (simplicial) spectrum of $r$-dimensional polyhedra. *In* ([2]) *it is shown that there exists a one-dimensional in the sense of* Ind *(i.e. also in the sense of* ind *and* dim*) bicompactum for which there is no one-dimensional polyhedral (i.e. also simplicial)* ***spectrum approximating it***. Further, it is known ([5]) that every bicompactum is the limit of a polyhedral spectrum with projections "into," but it was not clear whether every bicompactum is the limit of a spectrum of polyhedra with projections "onto" ****. Below examples of bicompacta $AP$ and $AP'$ will be constructed which give a negative answer to the question posed; moreover $\dim AP = 1$, $\operatorname{ind} AP \geq 2$, while $\dim AP' = \operatorname{ind} AP' = \operatorname{Ind} AP' = 1$. The construction of the bicompacta $AP$ and $AP'$ simultaneously answers negatively the question, also posed by P. S. Aleksandrov, concerning the validity of the "sum theorem" (for a countable number of summands) for polyhedral (simplicial) spectra, namely: the bicompactum $AP$ ($AP'$), which is not the limit of any one-dimensional, nor in general of any polyhedral spectrum (with projections "onto" ), can be represented as the sum of a countable number of bicompacta $Y_i$ ($Y_i'$), $i = 1, 2, ...$, which are limits of one-dimensional simplicial spectra*****. If spectra with projections "into" are taken, then the failure of the sum theorem for polyhedral and simplicial spectra follows from the fact that $\dim AP = 1$, while $\operatorname{ind} AP = 2$, i.e. $AP$ cannot be the limit of any one-dimensional polyhedral spectrum ([7]).

We shall construct the bicompactum $AP$ as the limit of a transfinite ordered spectrum of bicompacta $X_\alpha$. Take the set $T$ of transfinite numbers $\alpha \leq \omega_c$, where $\omega_c$ is the first ordinal number of cardinality continuum, and first divide

$T$ into continuum many pairwise disjoint sets $T_\theta$ of cardinality continuum, and then divide each $T_\theta$ again into continuum many pairwise disjoint sets $T_{\theta\nu}$ of cardinality continuum. Establish a one-to-one correspondence between the indices $\theta$ and all numbers $c_0$ from the interval $[0,1]$, and then do the same with the indices $\theta\nu$, for fixed $\theta$, and sequences of rational numbers $c_n$, $n = 1, 2, ...$,

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* The fact that every $r$-dimensional compactum in the sense of dim is the limit of a spectrum of $r$-dimensional compacta was proved by S. Mardešić in ([4]).

** Even with projections "into."

*** However, this bicompactum is the limit of a simplicial (i.e. also polyhedral) spectrum of two-dimensional polyhedra with projections "onto."

**** This question was posed by P. S. Aleksandrov.

***** I.e. $\dim Y_i = \operatorname{ind} Y_i = \operatorname{Ind} Y_i = 1$, whence it follows that the sum theorem for the dimensions ind and Ind is false even for summands that are limits of simplicial spectra of the corresponding dimension.

converging to $c_0$, with $c_n \neq c_m$ for $n \neq m$ and $c_n \neq c_0$. Thus, to each number $\alpha$ from $T_\theta$ there naturally corresponds a certain pair $(c_0, \{c_n\})$, $n = 1, 2, ....$ To distinguish the pairs corresponding to distinct $\alpha$, we shall mark them with the corresponding index: to the number $\alpha$ there corresponds the pair $(c_0^\alpha, \{c_n^\alpha\})$, $n = 1, 2, ....$ Denote the set $\bigcup_{n=0}^{\infty} c_n^\alpha$ by $l_\alpha$.

Construction of a spectrum for $AP$. As $X_1$ take the "vertical" segment $I_1 = \{(1, y),\ 0 \le y \le 1\}$. Suppose that, for all numbers $\alpha$ and $\alpha' \in T$ and $< \beta$, the spaces $X_\alpha$ and the projections $\mathfrak{D}_\alpha^{\alpha'}$ have been constructed.

1) Let the number $\beta - 1$ exist. Put

$$X_\beta = X_{\beta-1} \cup L_\beta \cup I_\beta,$$

where

$$I_\beta = \{(\beta, y),\ 0 \le y \le 1\}$$

is the "vertical"segment, and $L_\beta$ is obtained by multiplying the "horizontal" segment

$$J_\beta = \{x_\beta,\ 0 \le x_\beta \le 1\}$$

by the set $l_\beta$. Thus each point of $L_\beta$ receives two coordinates: $x_\beta$ and $c_n^\beta$. The sets $X_{\beta-1}$, $L_\beta$, and $I_\beta$ are regarded as open in $X_\beta$. The projection $\mathfrak{D}_{\beta-1}^\beta$ is defined as follows:

$$\mathfrak{D}_{\beta-1}^\beta(x_\beta, c_n^\beta) = (\beta - 1, c_n^\beta), \qquad \mathfrak{D}_{\beta-1}^\beta(\beta, y) = (\beta - 1, y),$$

and on $X_{\beta-1}$ it is the identity.

2) Suppose that $\beta$ is a limit number. Then $X_\beta$ is the limit of the spectrum

$$S_\beta = \{X_\alpha, \mathfrak{D}_\alpha^{\alpha'}\}, \qquad \alpha < \beta,$$

and, as is not hard to see,

$$X_\beta = \bigcup_{\alpha<\beta} X_\alpha \cup I_\beta,$$

where $X_\alpha \subset X_{\alpha'}$ when $\alpha < \alpha'$, and every neighborhood of the segment

$$I_\beta = \{(\beta, y),\, 0 \leq y \leq 1\}$$

contains all the sets $X_{\alpha'} \setminus X_\alpha$ for some fixed $\alpha$ and $\alpha' > \alpha$. We have

$$\mathfrak{D}_\alpha^\beta(\beta, y) = (\alpha, y);$$

for the remaining points, $\mathfrak{D}_\alpha^\beta$ coincides with some (and hence with all subsequent) projection $\mathfrak{D}_\alpha^{\alpha'}$. The bicompactum $AP$ is obtained as the limit of the spectrum

$$S = \{X_\alpha, \mathfrak{D}_\alpha^\beta\}, \qquad \alpha < \omega_c.$$

The bicompactum $AP'$ is a subset of $AP$ and is obtained by deleting from $AP$ all $L_\beta$, except for the subset $1_\beta \times l_\beta$. Since for every $\alpha$ we have $\dim X_\alpha = 1$, it follows that

$$\dim AP(AP') = 1.$$

We shall write $(x', y) < (x'', y)$ if: a) $x' = \alpha$, $x'' = \beta$, and $\beta > \alpha$; b) $x' \in L_\alpha$, $x'' \in L_\beta$, and $\beta > \alpha$; c) $x' \in L_\alpha$, $x'' = \alpha$; d) $x'$ and $x'' \in L_\beta$, i.e. $x' = x'_\beta$, $x'' = x''_\beta$, and $x'_\beta < x''_\beta$.

**Lemma 1.** Suppose we have a continuous mapping $f$ of the bicompactum $AP$ onto a polyhedron $P$. Then $f(x, y) = f(\omega_c, y)$, starting from some point $(\alpha, y)$.

**Lemma 2.** Under the conditions of the preceding lemma, for all $y$ there exists an $\alpha_0$ such that

$$f(x, y) = f(\omega_c, y)$$

for $x \geq \alpha_0$.

**Proof.** For all rational $y$, by Lemma 1 we find numbers $\alpha_y$, and, since the set of the numbers $\alpha_y$ is countable, there exists $\alpha_0 < \omega_c$ and $\geq \alpha_y$ for all rational $y$. This $\alpha_0$ will be the desired one, for $f$ is continuous and the set of points $(x, y)$ with rational second coordinates is, by construction, everywhere dense in $AP$, i.e., also in the set of points $(x, y)$ with $x \geq \alpha_0$.

We now show that $AP$ and $AP'$ will not be limits of any polyhedral spectra with projections "onto."

Suppose that the bicompactum $AP$ is represented as the limit of a polyhedral spectrum

$$S = \{P_\xi, \mathfrak{D}_\xi^\eta\}$$

with projections "onto." Take a polyhedron $P_\xi$ in which

$$\mathfrak{D}_\xi(\omega_c, 0) \neq \mathfrak{D}_\xi(\omega_c, 1).$$

Then on the segment

$$I_{\omega_c} = \{(\omega_c, y),\ 0 \leq y \leq 1\}$$

there will be found a point $(\omega_c, y_0)$ such that in every neighborhood of it, relative to the segment $I_{\omega_c}$, there is at least one point whose image in $P_\xi$ does not coincide with the image of the point $(\omega_c, y_0)$. If this is so, then there is a sequence of points $(\omega_c, y_n)$, $n = 1, 2, ...$, such that

$$\mathfrak{D}_\xi(\omega_c, y_n) \neq \mathfrak{D}_\xi(\omega_c, y_m) \neq \mathfrak{D}_\xi(\omega_c, y_0),$$

where the $y_n$ are rational and converge to $y_0$. Consider the pair $(c_0, \{c_n\})$ with $c_0 = y_0$ and $c_n = y_n$, $n = 1, 2, ....$ By Lemma 2, for $\mathfrak{D}_\xi$ we find $\alpha_0$. There exists such a $\beta_0 \geq \alpha_0$ that

$$(c_0^{\beta_0}, \{c_n^{\beta_0}\}) = (c_0, \{c_n\}),$$

for the indices $\beta$ to which the pair $(c_0, \{c_n\})$ corresponds form a continuum, i.e. they

occur arbitrarily far away. Now take in the spectrum $S$ a polyhedron $P_\eta$, $\eta > \xi$, in which

$$\mathfrak{F}_\eta(L_{\beta_0}) \cap \mathfrak{F}_\eta(AP \setminus L_{\beta_0}) = \Lambda,$$

i.e. the image $P_\eta^*$ of the set $L_{\beta_0}$ is open and closed in $P_\eta$ (such a polyhedron $P_\eta$ can always be found, since $L_{\beta_0}$ is open and closed in $AP$). The polyhedron $P_\eta^*$ must contain a countable number of pairwise disjoint open-and-closed sets

$$\mathfrak{F}_\eta(J_{\beta_0} \times c_n^{\beta_0}), \qquad n = 1, 2, \dots,$$

since

$$\mathfrak{F}_\xi(J_{\beta_0} + c_n^{\beta_0}) = \mathfrak{F}_\xi(\omega_c, y_n = c_n^{\beta_0}) \neq \mathfrak{F}_\xi(\omega_c, y_m = c_m^{\beta_0}) = \mathfrak{F}_\xi(J_{\beta_0} \times c_m^{\beta_0})$$

for $m \neq n$ and $\eta > \xi$. But, being a polyhedron, $P_\eta^*$ cannot contain a countable number of open-and-closed sets. We have arrived at a contradiction. In a completely analogous way the proof of the absence of a polyhedral spectrum with projections "onto" is also carried out for $AP'$. In fact, it is easy to see that we have proved even more, namely: *the bicompacts $AP$ and $AP'$ are not limits of any spectra of locally connected compacta with projections "onto."*

We now represent the bicompact $AP$ ($AP'$) as a countable sum of bicompacts $Y_i$ possessing one-dimensional simplicial spectra. Put

$$Y_i = \bigcup_{\alpha<\omega_c} I_\alpha \cup \bigcap_{\alpha<\omega_c} (J_\alpha \times c_i^\alpha), \qquad i = 0, 1, 2, \dots,$$

where the topology in $Y_i$ is induced by the bicompact $AP$. We shall construct a one-dimensional simplicial spectrum of polyhedra, for example, for $Y_0$. The indices of the required spectrum will be all possible finite sets $a = (\alpha_1, \dots, \alpha_s)$ of numbers $\alpha < \omega_c$, ordered by inclusion. We construct $P_a$, for example, for $a = (\alpha_1, \alpha_2)$. This will be the sum of intervals in the coordinate plane $xOy$:

$$I^a_{\alpha_1} = \{(x, y),\ x = 0,\ 0 \le y \le 1\},$$

$$J^a_{\alpha_1+1} = \{(x, y),\ 1 \le x \le 2,\ y = c_0^{\alpha_1+1}\},$$

$$I^a_{\alpha_2} = \{(x, y),\ x = 3,\ 0 \le y \le 1\},$$

$$J^a_{\alpha_2+1} = \{(x, y),\ 4 \le x \le 5,\ y = c_0^{\alpha_2+1}\}$$

and

$$I^a = \{(x, y),\ x = 6,\ 0 \le y \le 1\}.$$

Put

$$\mathfrak{F}_a(x, y) = \{(0, y) \text{ for } x \le \alpha_1;\ (x_{\alpha_1+1} + 1,\ c_0^{\alpha_1+1}) \text{ for } (x, y) \in J_{\alpha_1+1} \times c_0^{\alpha_1+1};$$

$$(3, y) \text{ for } \alpha_1 + 1 \le x \le \alpha_2;\ (x_{\alpha_2+1} + 4,\ c_0^{\alpha_2+1}) \text{ for } (x, y) \in J_{\alpha_2+1} \times c_0^{\alpha_2+1},$$

and, finally,

$$(6, y) \text{ for } x \ge \alpha_2 + 1\}.$$

The spectrum thus constructed will be simplicial and one-dimensional. In the case of the bicompact $AP'$ we proceed analogously.

It remains to show further that

$$\dim AP' = \operatorname{ind} AP' = \operatorname{Ind} AP' = 1$$

and that

$$\operatorname{ind} AP' \geq 2.$$

We first prove the second assertion, i.e. that $\operatorname{ind} AP \geq 2$. The proof is similar to those carried out in a like situation earlier (see, for example, ([6])). Take the point $M_0 = (\omega_c, 0)$ and consider its neighborhood $OM_0$, assuming that

$$[OM_0] \cap (E_1 = \{(x, y),\ y = 1\}) = \Lambda.$$

Let the point $(\omega_c, y_0)$ be the upper boundary of the points of $OM_0$ on the interval $I_{\omega_c}$. For each point $(\omega_c, y')$ of the set

$$O_{\omega_c} = OM_0 \cap I_{\omega_c}$$

one can find a basic neighborhood

$$O_{y'} = \{(x, y),\ x \geq \alpha_{y'},\ 0 \leq y'_1 \leq y \leq y'_2 < y_0\},$$

contained in $OM_0$. One can cover the whole set $O_{\omega_c}$ by a countable number of such neighborhoods $O_{y'_k}$, and take $\alpha_0 < \omega_c$ and $\geq$ all $\alpha_{y'_k}$. It is clear that for $x \geq \alpha_0$ and $y < y_0$ all points $(x, y)$ belong to $OM_0$, i.e. for $y = y_0$, for arbitrarily large $\alpha$, there exist intervals

$$J_\alpha \times (c_0^\alpha = y_0)$$

from $L_\alpha$ belonging to $[OM_c]$. Such $\alpha$ will be, for example, those in the corresponding pairs $(c_0^\alpha = y_0, \{c_n^\alpha\})$ for which $c_n^\alpha$ converge to $c_0^\alpha = y_0$ both from above and from below. It is precisely such $\alpha$ that we shall consider further. If at least one of the intervals

$$J_\alpha \times (c_0^\alpha = y_0)$$

belongs to

$$\operatorname{Fr} \rho OM = [OM_0] \setminus OM_0,$$

then $\operatorname{ind} \operatorname{Fr} \rho OM_0 \geq 1$. Suppose that no such interval belongs entirely to $\operatorname{Fr} \rho OM_0$. Then for all, i.e. for arbitrarily large, $\alpha$, to which pairs

$$(c_0^\alpha = y_0, \{c_n^\alpha\})$$

correspond (where $c_n^\alpha$ converge to $c_0^\alpha$ both from above and from below), on the intervals $J_\alpha \times y_0$ there is at least one point $M_\alpha = (x_\alpha, y_0)$ belonging to $OM_0$, i.e. $OM_0$ contains also some neighborhood $OM_{x_\alpha}$,

and hence also some sequence of points $(x_\alpha, y_k)$, $y_k = c_{n_k}^\alpha > y_0$. We now show that all points of some segment $\omega_c \times [y_0, y_1]$, $y_1 > y_0$, belong to the closure of $OM_0$. If there were no such segment, then there would exist a sequence of rational points $\{y_{2l}\}$, $l = 1, 2, ...$, $y_{2l} > y_0$ and $\neq y_{2m}$ for $m \neq l$, converging to $y_0$ and not belonging to $[OM_0]$. Take such a fixed pair $(c_0 = y_0, \{c_n\})^0$ that for $n = 2l$ we have $c_n = y_{2l}$, while for $n = 2l - 1$ all $y_n < y_0$. The indices $\beta$ for which $(c_0^\beta, \{c_n^\beta\}) = (c_0, \{c_n\})^0$ form a continuum; hence, as was shown above, $OM_0$ contains the points $(x_\beta, y_{2l})$ for $l \geq N_\beta$ for a continuum of numbers $\beta$, i.e., for a continuum of numbers $\beta$ the numbers $N_\beta$ coincide and are equal to $N$; that is, for arbitrarily large $\beta$, $OM_0$ contains the points $(x_\beta, y_{2l})$, $l \geq N$. It is now clear that the points $(\omega_c, y_{2l})$ for $l \geq N$ are contained in $[OM_0]$. We obtain a contradiction to the fact that $\Gamma p OM_0$ contains no segment $\omega_c \times [y_0, y_1]$. Thus, in every case $\operatorname{ind} \Gamma p OM_0 \geq 1$, i.e. $\operatorname{ind} AP$ and $\operatorname{Ind} AP \geq 2$. The fact that $\operatorname{ind} AP' = 1$ is sufficiently obvious, and then also $\operatorname{Ind} AP' = 1$. Thus, bicompacta that are not limits of polyhedral spectra (and even spectra of locally connected compacta) with "onto" projections may be both bicompacta with coinciding dimensions ind, Ind, and dim, and with noncoinciding ones.

The author expresses sincere gratitude to P. S. Aleksandrov for proposing the problems solved here.

Received18 VIII 1961

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*Note: Figure translations are in progress. See original paper for figures.*