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Abstract

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TOPOLOGICAL-METRIC PROPERTIES OF THE MULTIPLICATIVE STRUCTURE OF THE INTEGERS

(Presented by Academician I. M. Vinogradov on 30 VI 1961)

Let us consider an infinite table in which the divisors of the natural number n in the leftmost column are marked by small circles. We shall interpret each horizontal row as an infinite binary fraction

$$\|n\| = 0, \varphi_2 \varphi_3 \dots \varphi_m \dots,$$

where $\varphi_m = 0$ or $\varphi_m = 1$ according as n is divisible by m or not.

The set $\{\|n\|\}_1^\infty$ will be a bounded countable set on the number line; we denote its completion by Ψ. In the present note we study the structure of Ψ in connection with the arithmetic of polyadic numbers (see ([1–3])).

Notation (see ([1])). $\mathfrak{S} = \{x, y, \dots\}$ is the ring of polyadic numbers. $S = \{a, b, \dots\}$ is the ring of integers in the topology induced by $\mathfrak{S}$. $E = \{\varepsilon\}$ is the multiplicative group of divisors of unity from $\mathfrak{S}$. $a + (m)$ is the residue class mod m in $\mathfrak{S}$ containing a. For $x \in \mathfrak{S}$ we denote:

$$\psi(x) = \|x\| = \sum_{m=2}^{\infty} \frac{\varphi_m(x)}{2^{m-1}},$$

where $\varphi_m(x) = 0,\ x \equiv 0\ (m);\ \varphi_m(x) = 1,\ x \not\equiv 0\ (m)$. $\Psi = \{\|x\|\} = \psi(\mathfrak{S})^*$. $P = \{p\}$ is the set of prime numbers. $Q = \{q\}$ is the set of natural powers of prime numbers. $Q^{-1} = \{q^{-1}\}$ is the set of numbers reciprocal to natural powers of prime numbers.

n	1	2	3	4	5	6	7	8	9	10	...
1											...
2											...
3											...
4											...
5											...
6											...
7											...
8											...
9											...

n	1	2	3	4	5	6	7	8	9	10	...
10											...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Preliminary lemmas. Let $\|x\| \neq \|y\|$,

$$\|x\| = \sum_{m=2}^{\infty} \varphi_m(x)\, 2^{-m+1}, \qquad \|y\| = \sum_{m=2}^{\infty} \varphi_m(y)\, 2^{-m+1},$$

$\varphi_i(x) = \varphi_i(y)$ for $i < k$, and $\varphi_k(x) \neq \varphi_k(y)$. We denote k by $q(x, y)$.

By direct verification we establish that:

1. The function $q(x, y)$ has the following properties:

a) $q(x, y) \in Q$;

b) if $\|x\| > \|y\| > \|z\|$, then

$$q(x, z) = \min(q(x, y), q(y, z)).$$

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* This indeed holds, since the mapping $\psi(x) = \|x\|$, being a continuous mapping of the compact $\mathfrak{S}$ into $[0, 1]$, is closed.

Now set $\rho(x, y) = q^{-1}(x, y)$, if $\|x\| \neq \|y\|$, and $\rho(x, y) = 0$, if $\|x\| = \|y\|$. A consequence of item 1 is:

2) a) $\rho(x, y) \geq 0$; $\rho(x, y) = 0$ if and only if $\|x\| = \|y\|$;

b) $\rho(x, y) = \rho(y, x)$;

c) $\rho(x, y) \in Q^{-1}$, if $\|x\| \neq \|y\|$;

d) if $\|x\| \geq \|y\| \geq \|z\|$, then $\rho(x, z) = \max(\rho(x, y), \rho(y, z))$.

For a natural number $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $p_1^{a_1} < p_2^{a_2} < \cdots < p_k^{a_k}$, introduce the following notation: $q(n) = p_k^{a_k}$; $q^*(n)$ is the natural power of a prime following $q(n)$ in the natural ordering Q; $p(n) = p_k$;

$K(n) = [1, 2, \ldots, q(n)]$; $d(n) = \dfrac{n}{p_k}$.

If d is a natural divisor of x, we write $d \mid x$.

3. *Let* $d \mid K(n)$, $d_1 \mid K(n)$, $\|d\| \geq \|d_1\|$. *Then:*

a) $q^{-1}(n) \leq q^{-1}(d)$;

b) if $\rho(x, d) < q^{-1}(d)$, then $d \mid x$ and $\|d\| \geq \|x\|$, by item 2.1, ([1]);

c) $\rho(d_1, d) < q^{-1}(n)$ if and only if $d_1 = d$;

d) $\rho(d_1, d) = q^{-1}(n)$ if and only if $d_1 = dp(n)$ and $q(n) \mid d_1$.

A consequence of item 3 is:

4. $\|n\| = \max\{\|d_1\| : d_1 \mid K(n),\ \|d_1\| < \|d(n)\|\}$.

Indeed, $n = d(n)p(n)$. Therefore $\rho(n, d(n)) = q^{-1}(n)$, by item 3 d). Let $\|d(n)\| \geq \|d_1\| \geq \|n\|$. Then, by item 2 d), $\rho(d_1, d(n)) \leq q^{-1}(n)$, whence, by items 3 c) and d), either $d_1 = d(n)$ or $d_1 = n$.

For $\mathfrak{M} \subset \mathfrak{S}$ we shall denote by $\|\mathfrak{M}\|$ the set of elements $\{\|x\|\}$, $x \in \mathfrak{M}$.

Using items 2.6 and 2.9 of ([1]), it is easy to establish:

5. *Let $d \mid m$. The following conditions are equivalent:*

1) $\|x\| \in \|d + (m)\|$;

2) $x \equiv d\varepsilon\ (m)$ for some $\varepsilon \in E$;

3) $\|x + (m)\| = \|d + (m)\|$;

4) $x \in d\left\{\varepsilon + \left(\dfrac{m}{d}\right)\right\}$;

5) $(x, m) = d$.

An immediate consequence of Lemmas 1-5 is:

Main theorem. *Let $K = K(n)$, $d \mid K$, $d \neq K$, $\|d'\| = \max\{\|d_1\| : d_1 \mid K,\ \|d_1\| < \|d\|\}$.*

The following conditions are equivalent:

1) $\|x\| \in \|d + (K)\|$;

2) $x \equiv d\varepsilon\ (K)$;

3) $\|x + (K)\| = \|d + (K)\|$;

4) $x \in d\left\{\varepsilon + \left(\dfrac{K}{d}\right)\right\}$;

5) $(x, K) = d$;

6) $\rho(x, d) < q^{-1}(n)$;

7) $\rho(x, d) \leq q_*^{-1}(n)$;

8) $\|d\| \geq \|x\| > \|d'\|$;

9) $\|d\| \geq \|x\| \geq \|d\gamma\|$

for some γ, relatively prime to $\dfrac{K}{d}$, and some $\varepsilon \in E$. In the case $d = K$, conditions 8) and 9) are replaced by the condition $\|d\| \geq \|x\|$.

Set, in accordance with item 2.2, § 2 ([1]): $p^{\vartheta(0)} = \lim\limits_{k\to\infty} p^{k!}$ in $\mathfrak{S}$.

6. *Let* $d \mid K$, $K = K(n)$, *and* $\alpha = \prod\limits_{p \nmid K/d} p^{\vartheta(0)}$. *Then* $\|\alpha\| = \inf \|d + (K)\|$.

Indeed, $(\alpha, K) = d$, since $\left(\dfrac{\alpha}{d}, \dfrac{K}{d}\right) = 1$. Therefore $\|\alpha\| \in \|d + (K)\|$.

If $\|x\| \in \|d + (K)\|$, then $(x, K) = d$ and $x = dt$, where $\left(t, \frac{K}{d}\right) = 1$. Therefore x divides a and $\|x\| \geqslant \|a\|$.

Corollary of the main theorem. Since $(\alpha) = (\beta)$ if and only if $\|\alpha\| = \|\beta\|$ (see 1.5, § 2 (1)), $\|\alpha\|$ is a symbol of the principal ideal (α). In this sense the topology Ψ naturally determines a topologization of the set of principal ideals in $\mathfrak{S}$. It follows from the main theorem that the operations of taking the greatest common divisor (g.c.d.) and the least common multiple (l.c.m.) of two ideals are continuous in this topology. Thus we have the theorem:

Theorem 1. *Taking as a complete system of neighborhoods of a natural number* d *the collection of sets* $\{a : (a, m) = (d, m)\}_{m=1}^{\infty}$, *we turn the set of natural numbers into a topological structure with respect to the operations of taking g.c.d. and l.c.m. The resulting structure is metrizable, but not complete. Its completion is isomorphic to the topological structure of the set of closed (principal) ideals of the ring of polyadic numbers with respect to the operations of taking g.c.d. and l.c.m. The topological space of the latter structure is homeomorphic to* Ψ, *and this homeomorphism is established by assigning to the ideal* (α) *the number* $\|\alpha\|$.

Next, $\rho(x, y)$, by definition, depends only on $\|x\|$ and $\|y\|$. Therefore, by 2 and the main theorem:

7. $\rho(\|x\|, \|y\|) = \rho(x, y)$ metrizes Ψ.

The mapping ψ is continuous, since $\|x - y\|$ metrizes $\mathfrak{S}$. As a continuous mapping of a compact space into a Hausdorff space, ψ is a closed mapping. We obtain:

8. ψ is a continuous, closed, and open (by the main theorem) mapping of $\mathfrak{S}$ onto Ψ.

Using 6 and the main theorem, it is easy to establish:

Theorem 2. Ψ *is a perfect nowhere dense set on the number line.* $\|n\|$, *where* $n \geqslant 2$, *is the left endpoint of an interval contiguous to* Ψ, *whose right endpoint is* $\|\alpha(n)\|$, *where*

$$\alpha(n) = d(n) \prod_{p \times \frac{K(n)}{d(n)}} p^{\theta(0)}.$$

There are no other intervals contiguous to Ψ*.*

A proof of this theorem, based on other foundations, was given by the author in paper [2].

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CITED LITERATURE

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[3] E. V. Novoselov, *Izv. Vyssh. uchebn. zaved.*, Mathematics, No. 3 (22), 66 (1961).

Note: Figure translations are in progress. See original paper for figures.