ON BOUNDARY VALUE PROBLEMS FOR EQUATIONS OF PARABOLIC TYPE WITH DISCONTINUOUS COEFFICIENTS

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Abstract

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ON BOUNDARY VALUE PROBLEMS FOR EQUATIONS OF PARABOLIC TYPE WITH DISCONTINUOUS COEFFICIENTS

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In the works of O. A. Oleinik (^{1,2}), the solution of boundary value problems for elliptic and parabolic equations of the second order with discontinuous coefficients was obtained as the limit of solutions of the corresponding boundary value problems for equations with smooth coefficients. Such an approach, based on a certain smoothing of the coefficients of the equation, the use of a priori estimates, and the embedding theorems of S. L. Sobolev (^{3}) and S. M. Nikol' skii (^{4}), makes it possible to obtain the solution of boundary value problems for a broad class of conjugation conditions on the surfaces of discontinuity of the coefficients and to investigate the smoothness of the solution obtained.

In the present note, by applying the methods developed in (^{1,2}), the solution of certain boundary value problems for parabolic equations of the second order with discontinuous coefficients is obtained.

1. Let a bounded domain D of the space $x = (x_1, \dots, x_n)$ be divided into a finite number of subdomains D_r $(r = 1, 2, \dots, m)$ by $(n-1)$-dimensional smooth nonintersecting surfaces. Denote by Γ the boundary of the domain D; by Γ_{ik} the boundary separating the domains D_i and D_k $(i, k = 1, 2, \dots, m)$; $S = \Gamma \times [0, T]$, $Q_r = D_r \times (0, T]$, $r = 1, 2, \dots, m$, $S_{ik} = \Gamma_{ik} \times [0, T]$. In the cylinder $Q = D \times (0, T]$ consider the parabolic equation

$$A_0(x,t)\frac{\partial U}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x,t)\frac{\partial U}{\partial x_j}\right) + \sum_{i=1}^{n} B_i(x,t)\frac{\partial U}{\partial x_i} + C(x,t)U + F(x,t), \tag{1}$$

whose coefficients are sufficiently smooth functions in $\overline{Q} - \sum S_{ik}$ and have discontinuities of the first kind at the points of S_{ik}. Let

$$A_0 \geq \lambda_0 > 0, \qquad A_{ij} = A_{ji}, \qquad \sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j \geq \lambda \sum_{i=1}^{n} \xi_i^2, \qquad \lambda > 0, \qquad C(x,t) \leq 0. \tag{2}$$

We shall assume that in a neighborhood of any point of the surface Γ_{ik} one can introduce local coordinates $y_i = z_i(x_1, \dots, x_n)$, $i = 1, 2, \dots, n$, under which Γ_{ik} passes into the plane $y_n = 0$, where the $z_i(x_1, \dots, x_n)$ are sufficiently smooth functions.

We seek a function $U(x, t)$, continuous in Q, which satisfies equation (1) at all points of $Q - \sum S_{ik}$, the boundary conditions

$$U\big|_S = 0, \qquad U\big|_{t=0} = 0 \quad \text{on } D - \sum \Gamma_{ik} \tag{3}$$

and conjugation conditions of the form

$$l_{pq} U \equiv K_p \frac{dU}{dN_p} + K_q \frac{dU}{dN_q} + \delta_{pq} U = \varkappa_{pq} \quad \text{on } S_{pq}, \tag{4}$$

where $K_i \geq K_0 > 0$, $\delta_{pq} \geq 0$, and $\varkappa_{pq}$ are known functions on S_{pq},

$$\frac{d}{dN_p} = \sum_{i,j=1}^{n} A_{ij}^p(x, t) \cos(\nu_p, x_i) \frac{\partial}{\partial x_j},$$

ν_p is the normal to S_{pq}, exterior with respect to the domain Q_p, A_{ij}^p is the limiting value of A_{ij} on the boundary of the domain Q_p.

We shall consider only those solutions $U(x, t)$ of this problem which have derivatives $\partial U / \partial x_i \in L_2(Q)$, $i = 1, 2, \dots, n$. In what follows we use the notation adopted in (5).

Theorem 1. Let Γ belong to the class $A^{(2)}$, S_{pq} to the class $A^{(l+2)}$; the functions

$$A_{ij} \in C^{(l+1)}(\overline{Q}_r), \quad B_i,\ C,\ A_0,\ F \in C^{(l)}(\overline{Q}_r) \quad (r = 1, 2, \dots, m),$$

$$K_p \in C^{(l+1)}(\overline{S}_{pq}), \quad \delta_{pq}, \varkappa_{pq} \in C^{(l)}(\overline{S}_{pq}).$$

Then, for $l > n + k + 1$ and $k \geq 2$, there exists a unique solution $U(x, t)$ of the problem (1), (3), (4). This solution belongs to the class $C^{(k)}(\overline{Q} \cap \overline{Q}_r)$, $r = 1, 2, \dots, m$.

We shall obtain the solution of the problem (1), (3), (4) in the following way. Cover the domain D by a finite number of sufficiently small domains ω_k, $k = 1, 2, \dots, N$, such that in a domain ω_k containing points of the surface Γ_{pq} it is possible to pass to local coordinates $y_1^k, \dots, y_n^k$ in such a way that $y_n^k = y_n^l$ in the intersection of ω_k and ω_l. Let $e_k(x, t)$, $k = 1, 2, \dots, N$, be infinitely differentiable functions, equal to zero outside $\omega_k \times [0, T]$ and in some neighborhood of its boundary, and such that everywhere in $\overline{Q}$

$$0 \leq e_k(x, t) \leq 1, \qquad \sum_{k=1}^{n} e_k(x, t) \equiv 1.$$

By $\varkappa_r$ and δ_r denote smooth functions in $\overline{Q}_r$ such that $\varkappa_p - \varkappa_q = \varkappa_{pq}$, $\delta_p - \delta_q = \delta_{pq}$ on the surface S_{pq}. Let $K_r(x,t)$ be a smooth function in $\overline{Q}_r$, coinciding with K_r on the boundary of Q_r. Denote by $K(x,t)$ the function in Q equal to $K_r(x,t)$ in the domain Q_r. Define $\varkappa$ and δ analogously.

Construct the following functions:

$$\varkappa_k^h = e_k(\varkappa)_k^h, \qquad \delta_k^h = e_k(\delta)_k^h, \qquad k = 1, 2, \dots, N;$$

$$A_0^h = \sum_{k=1}^N e_k(KA_0)_r^h, \qquad A_{ij}^h = \sum_{k=1}^N e_k(KA_{ij})_k^h,$$

$$B_i^h = \sum_{k=1}^N e_k \left(KB_i - \sum_{j=1}^N \frac{\partial K}{\partial x_j} A_{ij} \right)_k^h, \qquad C^h = \sum_{k=1}^N e_k(KC)_k^h - \sum_{k=1}^N \left(\frac{\partial(\delta e_k)}{\partial y_n^k} \right)_k^h,$$

$$F^h = \sum_{k=1}^N e_k(KF)_k^h - \sum_{k=1}^N \left(\frac{\partial(\varkappa e_k)}{\partial y_n^k} \right)_k^h,$$

where $(\theta)_k^h$ denotes the averaging (see (3)) of the function θ in the domain $\omega_k \times [0,T]$, with averaging radius h with respect to the variables $y_1^k, \dots, y_n^k, t$.

Consider the sequence of equations

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij}^h \frac{\partial U_h}{\partial x_j} \right) + \sum_{i=1}^n B_i^h \frac{\partial U_h}{\partial x_i} + C_h^{hU} + \sum_{k=1}^N \frac{\partial \delta_k^h}{\partial y_n^k} U_h$$

$$= A_0^h \frac{\partial U_h}{\partial t} - F^h + \sum_{k=1}^N \frac{\partial \varkappa_k^h}{\partial y_n^k}, \qquad 0 < h < 1. \tag{5}$$

Transforming in (5) the terms containing derivatives with respect to $y_n^1, \dots, y_n^N$, we obtain

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij}^h \frac{\partial U_h}{\partial x_j} \right) + \sum_{i=1}^n B_i^h \frac{\partial U_h}{\partial x_i} + C_h^{hU} + \sum_{k=1}^N \sum_{i=1}^n \mu_{ik} \frac{\partial \delta_k^h}{\partial x_i} U_h$$

$$= A_0^h \frac{\partial U_h}{\partial t} - F^h + \sum_{k=1}^N \sum_{i=1}^n \mu_{ik} \frac{\partial \varkappa_k^h}{\partial x_i}, \tag{6}$$

where μ_{ik} are some smooth functions.

We shall obtain the solution of problem (1), (3), (4) as the limit, as $h \to 0$, of the solutions of equations (6) under the boundary conditions

$$U_h\big|_{t=0} = 0, \qquad U_h\big|_S = 0. \tag{7}$$

It is easy to show that U_h and $\partial U_h/\partial x_i$ are uniformly bounded in the norm $L_2(Q)$. Further, analogously to how this was done in (^{2}), we establish integral estimates for derivatives of higher order and, using the theorems of S. L. Sobolev (^{3}) and S. M. Nikol' skii (^{4}), obtain the existence of a limit $U(x,t)$ for the sequence U_h as $h \to 0$, possessing the differentiability properties indicated in Theorem 1.

The fulfillment of the conjugation condition is proved in the same way as in (^{2}). The continuity of $U(x,t)$ at the points of S and $D - \sum \Gamma_{ik}$ is established with the aid of barriers. The uniqueness of the solution of problem (1), (3), (4) is established by multiplying the equation satisfied by the difference of two solutions of the problem U_1 and U_2 by the function $(U_1 - U_2)\psi_\varepsilon$, where ψ_ε denotes a function equal to zero in an ε-neighborhood of the boundary S and to one outside a 2ε-neighborhood of this boundary, and integrating the resulting equality over the domain Q. Integrating by parts and letting ε tend to zero, we obtain $U_1 \equiv U_2$.

Let us note that, analogously, one can obtain the existence and uniqueness of the solution of boundary-value problems for general elliptic and hyperbolic equations of second order with discontinuous coefficients under conjugation conditions of the form (4) on the surfaces of discontinuity of the coefficients.

2. Consider the linear parabolic equation

$$\frac{\partial U}{\partial t} = \sum_{i,j=1}^{n} A_{ij}(x,t)\frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(x,t)\frac{\partial U}{\partial x_i} + C(x,t)U + F(x,t) \tag{8}$$

in the cylinder $Q = D \times (0,T]$. Let $A_{ij}(x,t)$, $B_i(x,t)$, $C(x,t)$, and $F(x,t)$ be smooth functions everywhere in the domain Q, with the exception of a finite number of smooth surfaces S_r $(r = 1,2,\dots,m)$, where these functions have discontinuities of the first kind. Let the S_r divide Q into $m+1$ domains Q_r, and suppose that condition (2) is fulfilled in Q.

Assume that in some neighborhood of any point there is a possible nonsingular change of coordinates

$$\tau = t, \qquad y_i = y_i(x_1,\dots,x_n,t), \qquad i = 1,2,\dots,n, \tag{9}$$

under which the surface of discontinuity of the coefficients is transformed into a piece of the plane $y_n = 0$.

We shall seek a continuous function $U(x,t)$ in $\overline{Q}$, having continuous first derivatives in Q, satisfying equation (8) in $Q - \sum_{i=1}^{N} S_i$ and the boundary conditions

$$U\big|_{t=0} = \varphi, \qquad U\big|_{S} = \psi, \tag{10}$$

where φ and ψ are arbitrary continuous functions such that $\psi|_{t=0} = \varphi|_{\Gamma}$. We note that this problem was considered by I. V. Girsanov in $(^6)$ in the case when the coefficients A_{ij}, B_i and the surfaces of discontinuity S_r do not depend on t.

Theorem 2. Let the functions A_{ij}, B_i, C, F belong to $C^{(0,\mu)}(\overline{Q}_r)$, $0 < \mu < 1$, $r = 1, 2, \dots, m+1$, and to the class $C^{(l)}$ in some neighborhood of S_r, $l \geq n+1$, with $S_r \in A^{(l+2)}$, and let the surface Γ be such that each of its points can be touched by some ball lying outside Q. Then there exists in Q a unique solution of problem (8), (10).

The solution of problem (8), (10) can be obtained as the limit, as $h \to 0$, of solutions of the equations

$$\frac{\partial U_h}{\partial t} = \sum_{i,j=1}^{n} A_{ij}^h \frac{\partial^2 U_h}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} B_i^h \frac{\partial U_h}{\partial x_i} + C^h U_h + F^h \tag{11}$$

with boundary conditions

$$U_h\big|_{t=0} = \varphi, \qquad U_h\big|_S = \psi. \tag{12}$$

Here

$$A_{ij}^h = \sum_{k=1}^{N_0} e_k (A_{ij})_k^h; \quad B_i^h = \sum_{k=1}^{N_0} e_k (B_i)_k^h; \quad C^h = \sum_{k=1}^{N_0} e_k (C)_k^h \quad \text{and} \quad F^h = \sum_{k=1}^{N_0} e_k (F)_k^h,$$

e_k is an infinitely differentiable function equal to zero outside Ω_k ($k = 1, 2, \dots, N_0$);

$$\sum_{k=1}^{N_0} \Omega_k = Q; \qquad 0 \leq e_k(x,t) \leq 1; \qquad \sum_{k=1}^{N_0} e_k = 1,$$

and the domains Ω_k are so small that in each of them a change of coordinates of the form (9) is possible, with $y_n^k = y_n^l$ in the intersection of Ω_k and Ω_l.

It follows from the maximum principle that the U_h are uniformly bounded with respect to h in Q. Applying the a priori estimates of work (7), we obtain compactness, in the sense of uniform convergence of U_h and of all derivatives entering equation (11), in any closed domain $Q' \subset Q$ not containing points of S_r. Next, considering equation (11) in local coordinates $y_1, y_2, \dots, y_n, \tau$ in a neighborhood of a point $P_0 \in S_r$, we establish a priori estimates for derivatives of U_h up to order l, involving no more than two differentiations with respect to y_n. Applying the theorems of S. M. Nikol' skii (4), we obtain that the limiting function $U(x,t)$ for U_h as $h \to 0$ satisfies the required conditions in Q.

The fulfillment of conditions (10) is proved with the aid of barriers. The uniqueness of the solution of problem (8), (10) follows from work (8).

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Note: Figure translations are in progress. See original paper for figures.