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Abstract

Full Text

MATHEMATICS

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ON THE ORDER AND ON THE ASYMPTOTIC VALUE OF AN APPROXIMATION OF A NONDIFFERENTIABLE FUNCTION BY POSITIVE LINEAR OPERATORS OF A CERTAIN TYPE

(Presented by Academician S. L. Sobolev on 28 V 1962)

1. Let $\{\varphi_k(r)\}_1^\infty$ be a sequence of functions defined on some set $E = \{r\}$, and let r_0 be a limit point of this set.

Assume that the series $\sum_{k=1}^\infty \varphi_k(r)$ converges absolutely on E and

$$u_r(t) = \frac{1}{2} + \sum_{k=1}^\infty \varphi_k(r) \cos kt \geq 0 \quad (r \in E, -\pi \leq t \leq \pi).$$

Put

$$\mathcal{L}_r(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) u_r(t) dt \tag{1}$$

for every 2π -periodic function $f(x)$.

It is known ⁽⁴⁾ that if $\lim_{r \rightarrow r_0} \varphi_1(r) = 1$, then

$$\lim_{r \rightarrow r_0} \mathcal{L}_r(f; x) = f(x)$$

at every point $t = x$ of continuity of the function $f(x)$. Moreover, it is necessary that

$$\lim_{r \rightarrow r_0} \varphi_k(r) = 1 \quad (k = 1, 2, \dots).$$

Consequently,

$$\alpha(r) = \sum_{k=1}^{\infty} \frac{1 - \varphi_{2k}(r)}{4k^2 - 1}, \quad \beta(r) = \sum_{k=0}^{\infty} \frac{1 - \varphi_{2k+1}(r)}{(2k + 1)^2}$$

tend to zero as $r \rightarrow r_0$.

From a theorem of P. P. Korovkin (⁴, Theorem 15) it follows that the conditions

$$\lim_{r \rightarrow r_0} \frac{1 - \varphi_1(r)}{\alpha(r)} = 0, \quad \lim_{r \rightarrow r_0} \frac{1 - \varphi_1(r)}{\beta(r)} = 0$$

are equivalent. In this case

$$\lim_{r \rightarrow r} \frac{\alpha(r)}{\beta(r)} = 1.$$

If

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

is the Fourier series of the function $f(x)$, then the linear operator (1) can be represented in the form

$$\mathcal{L}_r(f; x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \varphi_k(r) (a_k \cos kx + b_k \sin kx).$$

This shows that the approximation of functions by the linear operators (1) can be regarded as a positive method of summation of Fourier series. In this direction one should note the general results of P. P. Korovkin (¹⁻⁴), in particular, his work (⁷), where he establishes orders of approximation of classes of functions by certain operators of the form (1).

In the present note the orders and asymptotic values of approximations of functions by the positive operators (1) are studied.

2. It is easy to prove that P. P. Korovkin's theorem (³) on the asymptotic value of the approximation of a twice differentiable function by linear positive polynomial operators of a certain type is also valid for approximation by the more general operators (1).

Theorem 1 (P. P. Korovkin). *In order that the equality*

$$\lim_{r \rightarrow r_0} \frac{\mathcal{L}_r(f; x) - f(x)}{\mathcal{L}_r(\psi; x) - \psi(x)} = \frac{D_2 f(x)}{D_2 \psi(x)}, \quad (2)$$

hold, it is necessary and sufficient that the equality

$$\lim_{r \rightarrow r_0} \frac{1 - \varphi_2(r)}{1 - \varphi_1(r)} = 4 \quad (3)$$

hold, where $D_2f(x)$ is the second generalized derivative of the function $f(x)$ at the point x .

From this theorem, under condition (3), there follows the asymptotic equality

$$\mathcal{L}_r(f; x) - f(x) = [1 - \varphi_1(r)]D_2f(x) + o[1 - \varphi_1(r)]$$

for every function $f(x)$ having at the point $t = x$ the second generalized derivative $D_2f(x)$.

Theorem 2. Suppose that at the point x there exist the right and left derivatives $f'_+(x)$ and $f'_-(x)$ of the function $f(x) \in \mathcal{L}(-\pi, \pi)$.

If

$$\lim_{r \rightarrow r_0} \frac{1 - \varphi_1(r)}{\alpha(r)} = 0, \quad (4)$$

then the equality

$$\lim_{r \rightarrow r_0} \frac{\mathcal{L}_r(f; x) - f(x)}{\alpha(r)} = \frac{2}{\pi} [f'_+(x) - f'_-(x)] \quad (5)$$

is valid.

Corollary. In particular, if

$$\lim_{n \rightarrow \infty} \frac{1 - \rho_2^{(n)}}{1 - \rho_1^{(n)}} = 4,$$

then for the positive polynomial operator

$$\mathcal{K}_n(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x)W_n(t) dt,$$

where

$$W_n(t) = \frac{1}{2} + \sum_{k=1}^n \rho_k^{(n)} \cos kt \geq 0 \quad (-\pi \leq t \leq \pi),$$

the equality

$$\lim_{n \rightarrow \infty} p(n) [\mathcal{K}_n(f; x) - f(x)] = \frac{f'_+(x) - f'_-(x)}{\pi}, \quad (6)$$

is valid, where

$$\frac{1}{p(n)} = \frac{1}{2m+1} + (2m+1)(1 - \rho_1^{(n)}), \quad m = \left[\frac{n}{2} \right].$$

Let

$$\Phi_r(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) \Pi_r(t) dt \quad (7)$$

be the Poisson operator, where

$$\Pi_r(t) = \frac{1}{2} \frac{1-r^2}{1-2r \cos t + r^2}.$$

Theorem 3. If the function $f(x) \in \mathcal{L}(-\pi, \pi)$ has finite right and left derivatives $f'_+(x)$ and $f'_-(x)$ at the point $t = x$, then the equality

$$\lim_{r \rightarrow 1} \frac{\Phi_r(f; x) - f(x)}{(1-r) \ln \frac{1+r}{1-r}} = \frac{f'_+(x) - f'_-(x)}{\pi} \quad (8)$$

holds.

Theorem 3 makes it possible to establish the corresponding asymptotic equality for analytic functions.

Theorem 4. Let $F(z)$ be an analytic function of class H_1 in $|z| < 1$, and let $F(e^{i\varphi}) = \lambda(\varphi) + i\nu(\varphi)$ be its boundary value. If $\lambda(\varphi)$ and $\nu(\varphi)$ have finite left and right derivatives $\lambda'_\pm(\varphi)$ and $\nu'_\pm(\varphi)$ at the point $\varphi = \varphi_0$ ($z_0 = e^{i\varphi_0}$), then the equality

$$\lim_{r \rightarrow 1} \frac{F(z) - F(z_0)}{(1-r) \ln \frac{1+r}{1-r}} = \frac{F'_+(z_0) - F'_-(z_0)}{\pi}, \quad (9)$$

holds, where

$$F'_+(z_0) = \lambda'_+(\varphi_0) + i\nu'_+(\varphi_0), \quad F'_-(z_0) = \lambda'_-(\varphi_0) + i\nu'_-(\varphi_0)$$

and $z = re^{i\psi}$ ($0 < r < 1$).

3. Let us note the following application of Theorem 2. Let

$$\tau(r) = \sup_{f \in Z_1} \max_{-\pi \leq t \leq \pi} |\mathcal{L}_r(f; x) - f(x)|,$$

where Z_1 is the class of 2π -periodic functions $f(x)$ for which

$$|f(x+t) + f(x-t) - 2f(x)| \leq 2|t|.$$

Theorem 5. If condition (4) is satisfied, then the equality

$$\lim_{r \rightarrow r_0} \frac{\tau(r)}{\alpha(r)} = \frac{4}{\pi} \quad (10)$$

holds.

Corollary 1. For the Poisson operator (7) the equality

$$\lim_{r \rightarrow 1} \frac{\tau(r)}{(1-r) \ln \frac{1+r}{1-r}} = \frac{2}{\pi} \quad (11)$$

holds.

Equality (11) was proved by I. P. Natanson ⁽⁵⁾.

Corollary 2. For the operator of P. P. Korovkin ⁽⁷⁾

$$A_r(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) V_r(t) dt,$$

where

$$V_r(t) = \frac{1}{2} + \sum_{k=1}^{\infty} r^{k^2} \cos kt,$$

the equality

$$\lim_{r \rightarrow 1} \frac{\tau(r)}{\sqrt{1-r}} = \frac{2}{\sqrt{\pi}} \quad (12)$$

holds.

Equality (12) was proved by P. P. Korovkin ⁽⁷⁾.

4. Let $\mathcal{L}^p(-\pi, \pi)$ be the space of 2π -periodic functions $f(x)$ for which

$$\|f(x)\|_p = \left\{ \int_{-\pi}^{\pi} |f(x)|^p dx \right\}^{1/p} < \infty.$$

Theorem 6. For every function $f(x) \in \mathcal{L}^p(-\pi, \pi)$ the inequality

$$\|\mathcal{L}_r(f; x) - f(x)\|_p \leq \left[1 + \frac{\pi}{\sqrt{2}} \right] \omega_p(f; \sqrt{1 - \varphi_1(r)}) \quad (p \geq 1),$$

holds, where

$$\omega_p(f; \delta) = \sup_{|t| \leq \delta} \|f(x+t) - f(x)\|_p$$

is the integral modulus of continuity of the function $f(x)$ in $\mathcal{L}^p(-\pi, \pi)$.

Theorem 7. If $f(x)$ is differentiable on $[-\pi, \pi]$ and $f'(x) \in \mathcal{L}^p(-\pi, \pi)$, then the inequality

$$\|\mathcal{L}_r(f; x) - f(x)\|_p \leq \left[\frac{\pi^2}{2} + \frac{\pi}{\sqrt{2}} \sqrt{1 - \varphi_1(r)} \right] \omega_p[f'; (1 - \varphi_1(r))] \quad (p \geq 1).$$

holds.

Theorem 8. If $f''(x)$ exists on $[-\pi, \pi]$ and $f''(x) \in \mathcal{L}^p(-\pi, \pi)$, then the inequality

$$\begin{aligned} \|\mathcal{L}_r(f; x) - f(x)\|_p &\leq \frac{\pi^2 \|f''(x)\|_p}{4} [1 - \varphi_1(r)] + \\ &+ \frac{\pi^2}{4} \left[(1 - \varphi_1(r)) + \frac{\pi}{2} \sqrt{4 - \frac{1 - \varphi_2(r)}{1 - \varphi_1(r)}} \right] \omega_p[f''; (1 - \varphi_1(r))]. \end{aligned} \quad (13)$$

From inequality (13) it follows that condition (3) affects the rate of approximation of a twice differentiable function by the positive linear operators (1).

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