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Abstract

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ON INTEGRAL OPERATORS ACTING IN SPACES $\mathcal{L}_p$

(Presented by Academician S. L. Sobolev, May 19, 1962)

1. In the study of differential and integral equations and in a number of other questions, criteria for the continuity and complete continuity of a certain integral operator (linear or nonlinear) in one or another functional space play an important role. Many authors have dealt with proofs of various such criteria (see, for example, ([1–5])). The present paper is devoted to generalizations of certain known criteria and to a number of new criteria for the continuity and compactness of integral operators acting in the spaces $\mathcal{L}_p$. It is directly related to the paper ([6]).

The spaces $\mathcal{L}_p(\Omega)$ are understood in the usual sense; moreover, unless otherwise specified, it is assumed that p may take any positive (and even infinite) value. For $p < 1$ the spaces $\mathcal{L}_p$ cease to be Banach spaces; nevertheless we shall continue to speak of the "norm," defining it just as in the case $p \geq 1$, and using in both cases the same notation $\|x\|_p$. For more detail on the spaces $\mathcal{L}_p$ for $p < 1$, see ([7]).

With regard to the set Ω, we shall either assume that it has finite absolutely continuous measure, or assume that it has infinite measure and is the union of an increasing sequence of sets Ω_α $(\alpha = 1, 2, ...)$ of finite absolutely continuous measure. In particular, as Ω one may consider sets of finite-dimensional Euclidean space with Lebesgue measure. We shall have to consider simultaneously spaces $\mathcal{L}_p(\Omega^*)$ with some other set Ω^*, satisfying what was said above; the norm of a function $x(s)$ in such a space will be denoted by $\|x\|_p^*$. By p' we shall, as usual, denote the exponent conjugate to p, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$.

2. Let us first consider the linear integral operator

$$Au(s) = \int_\Omega K(s,t)x(t)\,dt \qquad (s \in \Omega^*). \tag{1}$$

For fixed values of s, the kernel $K(s,t)$ is a function of t only; therefore one may speak of its norm in the various spaces $\mathcal{L}_p(\Omega)$.

Theorem 1. Suppose the kernel of the operator (1) satisfies the condition

$$\|K(s,t)\|_p = \varphi(s) \in \mathcal{L}_q(\Omega^*), \tag{2}$$

where $p \geq 1$, and q is arbitrary. Then the operator (1) acts from $\mathcal{L}_{p'}(\Omega)$ into $\mathcal{L}_q(\Omega^*)$ and is continuous. Moreover, the norm of the operator satisfies the inequality

$$\|Ax(s)\|_q^* \leq \|\varphi(s)\|_q^* \|x(t)\|_{p'}. \tag{3}$$

Theorem 2. Suppose that under the conditions of Theorem 1, q and one of the numbers p, $\operatorname{mes}\Omega$ are simultaneously finite. Then the operator (1) acts from $\mathcal{L}_{p'}(\Omega)$ into $\mathcal{L}_q(\Omega^*)$ and is completely continuous.

Under the condition $\operatorname{mes}\Omega, \operatorname{mes}\Omega^* < \infty$, Theorem 2 is contained in (3).

There exist examples of noncompact operators satisfying the conditions of Theorem 2 for $q = \infty$ or for $p = \operatorname{mes}\Omega = \infty$.

3. More general criteria for the continuity and compactness of the operator (1) can be obtained by imposing on the kernel two restrictions of type (2). In doing so one must use the so-called interpolation theorem ([8,9]), valid in the spaces $\mathcal{L}_p$ $(p \geqslant 1)$ for an arbitrary linear operator:

Let a linear operator A act from $\mathcal{L}_{p_1}(\Omega)$ to $\mathcal{L}_{q_1}(\Omega^*)$ and be bounded; let it simultaneously act from $\mathcal{L}_{p_2}(\Omega)$ to $\mathcal{L}_{q_2}(\Omega^*)$ and also be bounded. Then for every τ $(0 < \tau < 1)$ it is bounded as an operator from $\mathcal{L}_p(\Omega)$ to $\mathcal{L}_q(\Omega)$, where

$$\frac{1}{p} = \frac{1-\tau}{p_1} + \frac{\tau}{p_2}, \qquad \frac{1}{q} = \frac{1-\tau}{q_1}. \tag{4}$$

If A is completely continuous as an operator from $\mathcal{L}_{p_1}(\Omega)$ to $\mathcal{L}_{q_1}(\Omega^*)$, then it acts from $\mathcal{L}_p(\Omega)$ to $\mathcal{L}_q(\Omega^*)$ also completely continuously.

The interpolation theorem also gives an estimate of the norm of the operator, asserting its logarithmic convexity in τ. It can be shown that under certain restrictions it is also valid for $p < 1$; in particular, it can be applied to integral operators with nonnegative kernels.

Let us formulate a theorem imposing restrictions of type (2) on the kernel itself and on its adjoint.

Theorem 3. *Let the kernel of the operator (1) satisfy simultaneously the two conditions*

$$\|K(s,t)\|_{p_1} = \varphi_1(s) \in \mathcal{L}_{q_1}(\Omega^*); \qquad \|K(s,t)\|_{p_2}^* = \varphi_2(t) \in \mathcal{L}_{q_2}(\Omega), \tag{5}$$

where $p_1, p_2, q_2 \geqslant 1$, and q_1 is arbitrary.

Then for every τ $(0<\tau<1)$ the operator (1) is continuous, and in the case of finite p_1, p_2, q_1, q_2 it is a completely continuous operator acting from $\mathcal{L}_p(\Omega)$ to $\mathcal{L}_q(\Omega^*)$, where p and q are determined from the equalities

$$\frac{1}{p}=\frac{1-\tau}{p_1'}+\frac{\tau}{q_2}, \qquad \frac{1}{q}=\frac{1-\tau}{q_1}+\frac{\tau}{p_2}. \tag{6}$$

Moreover, the norm of the operator (1) satisfies the inequality

$$\|A\| \leqslant \left(\|\varphi_1(s)\|_{q_1}^*\right)^{1-\tau}\left(\|\varphi_2(t)\|_{q_2}\right)^{\tau}. \tag{7}$$

The complete continuity of the operator A in Theorem 3 can also be asserted under some less stringent assumptions. Thus, if $q_1 \geqslant 1$, it is sufficient that the numbers of one of the pairs p_1, q_1 or p_2, q_2 be finite. Theorem 3 is especially convenient for operators with symmetric kernels, since for them only one condition of type (2) is sufficient.

4. Condition (2) for $p<1$, taken by itself, does not allow one to make any assertion about the operator (1). However, in combination with other conditions it can sometimes give some additional information about the operator, which will make it possible, as before, to apply the interpolation theorem.

Theorem 4. *Let the kernel of the operator (1) satisfy the conditions (5), where $0<p_1<1$, $p_2 \geqslant 1$, q_1 is arbitrary, and q_2 satisfies the inequality*

$$q_2 \geqslant \sigma=\frac{q_1 p_2(1-p_1)+p_1 p_2}{q_1(1-p_1)+p_1 p_2}. \tag{8}$$

Then for every τ $(0<\tau<1)$ the operator (2) acts from $\mathcal{L}_p(\Omega)$ to $\mathcal{L}_q(\Omega^*)$, where

$$\frac{1}{p}=\frac{1-\tau}{q_2'}+\frac{\tau(1-p_1)(q_2-\sigma)}{q_2(\sigma-p_1)}, \qquad \frac{1}{q}=\frac{1-\tau}{p_2}+\frac{\tau(1-p_1)}{\sigma-p_1}. \tag{9}$$

In this case the norm of the operator (1) satisfies the inequality

$$\|A\| \leq \left(\|\varphi_1(s)\|_{q_1}^*\right)^{\frac{\tau p_1(\sigma-1)}{\sigma-p_1}}\left(\|\varphi_2(t)\|_{q_2}\right)^{1-\frac{\tau p_1(\sigma-1)}{\sigma-p_1}}. \tag{10}$$

If q_1 is finite, or if the numbers of one of the pairs q_2, p_2 or $q_2, \operatorname{mes}\Omega^*$ are finite, then the operator (1) acts from $\mathcal{L}_p(\Omega)$ into $\mathcal{L}_q(\Omega^*)$ completely continuously.

To prove the theorem it is necessary first to show that A is continuous (completely continuous) as an operator from $\mathcal{L}_{p_0}(\Omega)$ into $\mathcal{L}_{q_0}(\Omega^*)$, where

$$p_0 = \frac{(\sigma - p_1)q_2}{(1-p_1)(q_2-\sigma)}, \qquad q_0 = \frac{\sigma - p_1}{1-p_1}, \tag{11}$$

and then to apply the interpolation theorem.

In the special case where $q_1 = q_2 = \infty$ and $\operatorname{mes}\Omega < \infty$, Theorem 4 was proved by L. V. Kantorovich ([3]) without using the interpolation theorem. Complete continuity of the operator A in this case may fail to hold.

Indeed, if as Ω one takes the interval $\left[0, \frac{1}{2^{p_1}-1}\right]$, and as Ω^* the interval $\left[0, \frac{1}{2^{p_2}-1}\right]$, and defines the linear integral operator with kernel

$$K(s,t) = 2^n \quad (n = 1, 2, \ldots), \quad \text{if } 2^{-np_1} < t \le 2^{(1-n)p_1}, \quad 2^{-np_2} < s \le 2^{(1-n)p_2},$$

and equal to zero for all other combinations of s, t, then the bounded set of functions in $\mathcal{L}_p(\Omega)$

$$x_k(t) = \begin{cases} 2^{kp_1/p}, & \text{for } 2^{-kp_1} < t \le 2^{(1-k)p_1}, \\ 0, & \text{for other values of } t \end{cases} \qquad (k = 1, 2, \ldots).$$

is mapped by it into a set of functions which is not compact in $\mathcal{L}_q(\Omega^*)$.

5. We now turn to the consideration of the nonlinear integral operator of P. S. Uryson

$$Bx(s) = \int_\Omega R[s, t, x(t)]\, dt \qquad (s \in \Omega^*), \tag{12}$$

where the function of three variables $R(s,t,x)$ is measurable in s, t for all x ($-\infty < x < \infty$) and continuous in x for almost all $\{s,t\} \in \Omega \times \Omega^*$. The following criterion for continuity and complete continuity of the operator (12) holds.

Theorem 5. Let the function $R(s,t,x)$ satisfy the inequality

$$|R(s,t,x)| \le K(s,t) f(t,x) \tag{13}$$

and let the operator $fx = f[t, x(t)]$ act from $\mathcal{L}_r(\Omega)$ into $\mathcal{L}_p(\Omega)$, while the operator (1) acts from $\mathcal{L}_p(\Omega)$ into $\mathcal{L}_q(\Omega^*)$ and is continuous (completely continuous), where $p > 1,\ q \ge 1,\ r$ is arbitrary.

Then the operator (12) acts from $\mathcal{L}_r(\Omega)$ into $\mathcal{L}_q(\Omega^*)$ and is also continuous (completely continuous).

Earlier ([6]) this criterion was obtained under the additional assumption that $\operatorname{mes}\Omega, \operatorname{mes}\Omega^* < \infty$. Approximating the operator (12) by operators B_α with kernels

$$R_\alpha(s,t,x) = \begin{cases} R(s,t,x), & \text{for } s \in \Omega^*_\alpha,\ t \in \Omega_\alpha, \\ 0, & \text{for other } s,t, \end{cases} \tag{14}$$

one can show its validity for arbitrary Ω and Ω^*.

In ([6]) it is shown that the operator (1), acting from $\mathcal{L}_p(\Omega)$ into $\mathcal{L}_q(\Omega^*)$ ($p > 1,\ q \geq 1$), is compact in measure, i.e. maps every set of functions bounded in $\mathcal{L}_p(\Omega)$ into a set of functions compact in measure. For the operator (12) this fact could not be proved in the general case.

However, an analogous assertion can be made for operators whose kernels satisfy the following condition:

A. For every $\varepsilon > 0$ there exists a set $\Delta_\varepsilon \subset \Omega^*$ such that $0 < \operatorname{mes}\Delta_\varepsilon < \varepsilon$ and, for all t and x, the inequality

$$|R(s_1,t,x)| \leq |R(s_2,t,x)|$$

holds as soon as $s_2 \in \Delta_\varepsilon,\ s_1 \in \Omega^* - \Delta_\varepsilon$.

Theorem 6. If $\operatorname{mes}\Omega,\ \operatorname{mes}\Omega^* < \infty$ and the operator (12) with kernel satisfying condition **A** acts from $\mathcal{L}_r(\Omega)$ to $\mathcal{L}_q(\Omega^*)$ ($0 < r, q < \infty$), then, as an operator on any $\mathcal{L}_p(\Omega)$ ($p > r$), it is compact in measure.

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Note: Figure translations are in progress. See original paper for figures.