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Abstract

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ON UNIQUENESS THEOREMS FOR THE SOLUTION OF CERTAIN BOUNDARY-VALUE PROBLEMS FOR EQUATIONS OF MIXED TYPE

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A problem analogous to those considered in the present article was first posed by F. I. Frankl in work ([1]), where he reduces the solution of the question of the flow around a wedge by a supersonic stream, in the case where a subsonic zone is formed in front of the wedge, to a boundary-value problem for Chaplygin' s equation in a certain domain of the hodograph plane. In ([1]) a uniqueness theorem was proved under a certain restriction, namely $\theta_0 < 54^\circ$ (where θ_0 is the angle between the side of the wedge and the direction of the incident stream).

In the present note the following are proved:

1. A uniqueness theorem for the equation

$$y\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad (1)$$

with boundary conditions: the values of the solution on one characteristic and on part of the contour in the elliptic half-plane are zero; on the remaining part of the contour in the elliptic half-plane the equality

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0;$$

holds; here a, b, and c are prescribed functions.

2. A uniqueness theorem for equation (1) under the same conditions, except for the last, which is replaced by

$$a\frac{\partial z}{\partial y} + b\frac{\partial z}{\partial y} = 0,$$

and for a certain particular contour.

3. A uniqueness theorem for the Lavrent' ev—Bitsadze equation

$$\frac{\partial^2 z}{\partial x^2} + \operatorname{sgn} y \frac{\partial^2 z}{\partial y^2} = 0 \tag{2}$$

under the boundary conditions of item 2.

The uniqueness theorems in all cases are proved without the restriction imposed by F. I. Frankl. From this one may conclude that the restriction was connected only with the choice of the method of proof.

§ 1. Let Δ be a simply connected domain of the plane (x, y), bounded by a piecewise smooth curve Γ in the half-plane $y > 0$, resting on the axis Ox at the points $O(0,0)$ and $C(m,0)$, and by two characteristics OB: $X - \frac{2}{3}(-y)^{3/2} = 0$ and CB: $X + \frac{2}{3}(-y)^{3/2} = m$ $(0 < m < 1)$. The curve Γ may go off to infinity.

Generalized Frankl problem. Find a solution of equation (1) in the domain Δ $(y \neq 0)$, continuous up to the boundary, bounded at infinity, having continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ in Δ, where near the points O and C they may tend to infinity of order less than 1, and satisfying the following conditions on the boundary of the domain:

on the characteristic OB: $z = 0$; on the contour Γ:

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0. \tag{3}$$

Here the coefficients a, b, and c may have discontinuities. In particular, on separate parts of the boundary Γ they may vanish. A special case of this problem is the previously mentioned problem on the flow around a wedge, which we shall call the Frankl problem.

Let Δ be a simply connected domain of the (x, y)-plane, bounded by two rays $OA : x = 0$ and $DE : x = 1$ $(y > 0)$, extending to infinity, and by the curve $CD : C(m, 0)$ and $D(1, y)$ in the half-plane $y > 0$, and by two characteristics OB and CB in the half-plane $y < 0$.

Frankl problem. Find a solution of equation (1) in the domain Δ $(y \neq 0)$, continuous up to the boundary, bounded at infinity, having continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ in Δ, where near the points O and C they may become infinite of order less than one, and satisfying on the boundary of the domain the conditions:

$$\begin{aligned} &\text{on the characteristic } OB && z = 0; \\ &\text{on the rays } OA \text{ and } DE && z = 0; \\ &\text{on } CD && a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 0. \end{aligned} \tag{3a}$$

In what follows we shall call the Frankl problem for the Lavrent' ev-Bitsadze equation the Frankl problem for equation (2).

§ 2. From the homogeneity of equations (1) and (2) and of the boundary conditions (3) and (3a) it follows that the solution of the problem is determined up to a constant factor. To remove the ambiguity (in this sense), we fix the solution at a certain point. Let $z(m,0)=1$. Assuming the existence of solutions of all the enumerated problems, we shall prove their uniqueness.

Theorem 1. *In the domain Δ there cannot exist more than one solution of the Frankl problem of the class indicated above.*

Proof. Transform the boundary condition (3a) on CD. Since

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\cos(\tau,x)+\frac{\partial z}{\partial y}\cos(\tau y),$$

$$\frac{\partial z}{\partial n}=\frac{\partial z}{\partial x}\cos(nx)+\frac{\partial z}{\partial y}\cos(ny),$$

where n is the exterior normal, then, taking into account the relation between the cosines of the introduced angles, we obtain the boundary condition in the form

$$[-a\cos(ny)+b\cos(nx)]\frac{\partial z}{\partial s}+[a\cos(nx)+b\cos(ny)]\frac{\partial z}{\partial n}=0. \tag{4}$$

In the elliptic part of the domain Δ $(y>0)$, the solution may attain a positive maximum (or a negative minimum) either on OC or on CD. But, as K. I. Babenko proved (3), on OC a positive maximum cannot be attained.

§ 3. We give the idea of the proof.*

Let $z(x,y)$ be a solution of the Tricomi problem, which is taken in characteristic coordinates ξ,η (here $\xi=x-\frac{2}{3}(-y)^{3/2}$ and $\eta=x+\frac{2}{3}(-y)^{3/2}$) in the hyperbolic part $(y<0)$ of the domain Δ. Introduce, as usual, the notation: $z(x,0)=\tau(x)$ and $z'_y(x,0)=\nu(x)$. Then, if on the characteristic $\xi=0$, $z(0,\eta)=0$, the identity holds

$$K'\int_0^{\xi}\tau(t)\,\frac{(\eta-\xi)^{2/3}\,dt}{(\eta-t)^{5/6}(\xi-t)^{1/6}}-K\int_0^{\xi}\nu(t)(\eta-t)^{-1/6}(\xi-t)^{-1/6}\,dt=0. \tag{5}$$

We obtain identity (5) by finding explicitly the solutions of two problems, subtracting them, and assuming that for $\xi=0$, $z=0$. The mentioned problems are:

1. Find a solution of equation (1) in the hyperbolic part of the domain Δ, if known are: its value τ for $y = 0$ and z for $\xi = 0$ (on the characteristic).
2. Find a solution of equation (1) in the hyperbolic part of the domain Δ, if known are: the values of its normal derivative ν for $y = 0$ and z for $\xi = 0$ (on the characteristic).

Let now $M = \max \tau(x)$, and let this maximum be attained at an interior point of the segment OC. Then the function

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* Work (3) has not been published; therefore we consider it necessary to give a brief account of the proof of this important theorem.

$$\zeta(\xi, \eta) = K' M \int_0^\xi \frac{(\eta - \xi)^{2/3}\, dt}{(\eta - t)^{5/6}(\xi - t)^{5/6}} - K \int_0^\xi \nu(t)(\eta - t)^{-1/6}(\xi - t)^{-1/6}\, dt \geq 0. \tag{6}$$

Denote the point at which the maximum of $\tau(x)$ is attained by x_0. Then, for $\xi = x_0$ and $\eta = x_0$, the equality

$$\zeta(x_0, x_0) = 0 \tag{7}$$

holds.

Indeed, assuming that $\tau(x)$ is a smooth function, we obtain

$$\tau(t) = \tau(\xi) + (t - \xi)\tau'[\xi + \theta(t - \xi)], \qquad 0 < \theta < 1.$$

Substituting this value of $\tau(t)$ into the first integral of identity (5), transforming it, and passing to the limit as $\eta \to \xi$, we obtain:

$$K'\tau(\xi)\frac{\Gamma(2/3)\Gamma(1/6)}{\Gamma(5/6)}.$$

Passing to the limit as $\eta \to \xi$ in the second term of identity (5), we have

$$K \int_0^\xi \nu(t)(\xi - t)^{-1/3}\, dt.$$

Thus, in the limit, identity (5) becomes

$$K' \frac{\Gamma(2/3)\Gamma(1/6)}{\Gamma(5/6)}\tau(\xi) - K \int_0^\xi \nu(t)(\xi - t)^{-1/3}\, dt = 0. \tag{8}$$

If we now put $\xi = x_0$, the left-hand side of the obtained identity turns into $\zeta(x_0, x_0)$, which proves the validity of equality (7).

Since at the point (x_0, x_0) the function $\zeta(x_0, x_0)$ attains a minimum, its derivative as $\eta \to x_0$ is nonnegative. Therefore (since $\eta > x_0$) the inequality

$$\lim_{\eta \to x_0} (\eta - x_0)^{1/3} \frac{\partial \zeta(x_0, \eta)}{\partial \eta} \geq 0 \tag{9}$$

holds.

Let us compute this same limit from expression (6). Denote

$$\zeta_1(x_0, \eta) = K' M \int_0^{x_0} \frac{(\eta - x_0)^{2/3}\, dt}{(\eta - t)^{5/6}(x_0 - t)^{5/6}},$$

$$\zeta_2(x_0, \eta) = K \int_0^{x_0} \nu(t)(\eta - t)^{-1/6}(x_0 - t)^{-1/6}\, dt.$$

After straightforward computations we obtain:

$$\lim_{\eta \to x_0} (\eta - x_0)^{1/3} \frac{\partial \zeta_1(x_0, \eta)}{\partial \eta} = -K' M x_0^{-2/3}. \tag{10}$$

For the second term,

$$(\eta - x_0)^{1/3} \frac{\partial \zeta_2}{\partial \eta} = -\frac{1}{6} K \int_0^{x_0} \nu(t) \frac{(\eta - x_0)^{1/3}\, dt}{(\eta - t)^{7/6}(x_0 - t)^{1/6}}. \tag{11}$$

Since $\nu(t)$ is continuous, represent it in the form
$\nu(t) = \nu(x_0) + \delta(x_0, t)$, where, as $t \to x_0$, $\delta \to 0$. Substituting $\nu(t)$ into (11), proving the smallness of the second term and computing the first integral, we obtain

$$\lim_{\eta \to x_0} (\eta - x_0)^{1/3} \frac{\partial \zeta_2}{\partial \eta} = -\overline{K} \nu(x_0), \qquad \text{where } \overline{K} = \frac{K\Gamma(5/6)\Gamma(1/3)}{6\Gamma(7/6)}.$$

Thus:

$$\lim_{\eta \to x_0} (\eta - x_0)^{1/3} \frac{\partial \zeta}{\partial \eta} = -K' M x_0^{-2/3} + \overline{K} \nu(x_0).$$

Applying Zaremba' s theorem ([4]) at the point x_0 (approaching the point from the elliptic half-plane) and taking into account the continuity of $v(x)$, we obtain $v(x_0) \leqslant 0$. Therefore

$$\lim_{\eta \to x_0} (\eta - x_0)^{1/3} \frac{\partial \xi}{\partial \eta} < 0. \tag{12}$$

Inequalities (9) and (12) are contradictory, which proves our assertion: at an interior point of the segment OC a positive maximum is not attained. For a negative minimum the arguments are analogous.

Thus, a positive maximum (or negative minimum) must be attained on CD.

§ 4. **Theorem.** *At an interior point of the arc CD, the solution of Frankl' s problem cannot attain a positive maximum (negative minimum).*

Suppose that at a point N of the arc CD a positive maximum is attained; then at this point $\partial z/\partial s = 0$, and the boundary condition (4) takes the form

$$[a\cos(nx) + b\cos(ny)]\frac{\partial z}{\partial n} = 0. \tag{13}$$

But, as follows from Zaremba' s theorem, at the point of a positive maximum $\partial z/\partial n < 0$. Hence, at the point N, $a\cos(nx) + b\cos(ny) = 0$. However, in gas dynamics ([1]) on CD the inequality $a\cos(nx) + b\cos(ny) \neq 0$ holds. Therefore a positive maximum cannot be attained at any interior point of CD. Consequently, it is attained either at the point D, or at the point C. However, by continuity, at the point D, $z = 0$. Therefore the positive maximum is attained at the point C. From this the uniqueness theorem for the solution of Frankl' s problem follows immediately.

Theorem 2. *In the domain Δ there cannot exist more than one solution of Frankl' s problem for the Lavrent' ev–Bitsadze equation of the same class as Frankl' s problem.*

The proof is entirely analogous to that given above.

Theorem 3. *In the domain Δ there cannot exist more than one solution of the generalized Frankl problem in the indicated class, if the condition*

$$[a\cos(nx) + b\cos(ny)]c < 0$$

is satisfied.

Proof. We transform condition (3) on parts of the contour Γ, analogously to how this was done for condition (3a) on CD. We obtain

$$[-a\cos(nx) + b\cos(ny)]\frac{\partial z}{\partial s} + [a\cos(nx) + b\cos(ny)]\frac{\partial z}{\partial n} + cz = 0. \tag{14}$$

If an interior point of the part of the contour Γ under consideration is a point of maximum of the solution, then at it $\partial z/\partial s = 0$, and from (14) it follows that

$$[a\cos(nx) + b\cos(ny)]\frac{\partial z}{\partial n} + cz = 0. \tag{15}$$

Since at a point of positive maximum $z > 0$ and $\partial z/\partial n < 0$, for condition (15) to be fulfilled it is necessary that

$$[a\cos(nx) + b\cos(ny)]c > 0.$$

If, however, the inequality

$$[a\cos(nx) + b\cos(ny)]c < 0, \tag{16}$$

holds, then the positive maximum will be attained at the point C. Thus, when inequality (16) is fulfilled, the uniqueness theorem also holds for the generalized Frankl problem.

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CITED LITERATURE

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[2] F. I. Frankl, *Prikl. matem. i mekh.*, **10**, issue 3, 421 (1946).
[3] K. I. Babenko, On the theory of equations of mixed type, Doctoral dissertation, Mathematical Institute named after V. A. Steklov, Academy of Sciences of the USSR, 1951.
[4] S. Zaremba, *UMN*, **1**, issue 3–4 (13–14), 125 (1946).

Note: Figure translations are in progress. See original paper for figures.