



Soviet-era science, translated into English

MATHEMATICS

V. M. VOLOSOV

1962

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196201.20451>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

V. M. VOLOSOV

AVERAGING IN SOME SYSTEMS OF DIFFERENTIAL EQUATIONS

(Presented by Academician I. G. Petrovskii, 12 III 1962)

In ^(1,2) averaging was studied in systems of the form

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon), \quad \dot{y} = Y(x, y, t, \varepsilon)$$

$$(x = \{x_1, \dots, x_n\}, y = \{y_1, \dots, y_m\}, X = \{X_1, \dots, X_n\}, Y = \{Y_1, \dots, Y_m\},$$

$\varepsilon > 0$ is a small parameter) under the assumption that, for certain functions, mean values exist along the integral curves of the degenerate system $x = \text{const}$, $\dot{y} = Y(x, y, t, 0) \equiv Y_0(x, y, t)$, and moreover these means do not depend on the initial values of the solutions of the degenerate system.

In the present paper averaging theorems are formulated in which dependence of the mean values on the choice of the trajectory of the degenerate system along which the averaging is performed is allowed.

We introduce the following requirements:

- 1) The function $X(x, y, t, \varepsilon)$ is defined and uniformly bounded for $0 \leq \varepsilon \leq \varepsilon_0$ ($\varepsilon_0 > 0$), $x, y, t \in G$ (G is some open, possibly unbounded, domain of the $(n + m + 1)$ -dimensional space x, y, t) and is uniformly continuous in ε with respect to the aggregate x, y, t .
- 2) $X_1(x, y, t) \equiv X|_{\varepsilon=0}$ is continuous, satisfies a Lipschitz condition in x, t , and has continuous bounded derivatives in y .
- 3) $Y(x, y, t, \varepsilon)$ is defined for $x, y, t \in G$, $0 \leq \varepsilon \leq \varepsilon_0$, is continuous, satisfies a Lipschitz condition in x, y, t , and has a partial derivative in ε that is continuous in ε uniformly with respect to x, y, t . $Y_1(x, y, t) \equiv \left. \frac{\partial Y}{\partial \varepsilon} \right|_{\varepsilon=0}$ is continuous, satisfies a Lipschitz condition in x, t , and has continuous bounded first-order derivatives in y .

- 4) Through every point $x_0, y_0, t_0 \in G$ there passes a unique integral curve $x = \text{const}$, $y = \varphi(x_0, y_0, t_0, t)$ of the degenerate system $x = \text{const}$, $\dot{y} = Y_0$, lying in G for all $t \geq t_0$, extendable to the boundary (or lying entirely inside) the domain G for $t \leq t_0$. The function φ is continuous and has continuous bounded derivatives with respect to y_0, t_0 , and

$$\sum_{i=1}^{m+1} D_i^2 \geq \text{const} > 0$$

everywhere in the domain G , where D_i are the minors of order m of the matrix

$$\left\| \frac{\partial \varphi}{\partial y_0}, \frac{\partial \varphi}{\partial t_0} \right\|.$$

- 5) In G there lies a smooth $(n+m)$ -dimensional manifold $M : x = a(\lambda)$, $y = b(\lambda)$, $t = c(\lambda)$, where $\lambda = \{\lambda_1, \dots, \lambda_{n+m}\}$ is a set of parameters ($\lambda \in \Lambda$, where Λ is an open domain, and for $\lambda \in \Lambda$ the point $a, b, c \in G$). The functions a, b, c are continuous and have continuous bounded derivatives. Moreover,

$$\sum_{i=1}^{n+m+1} A_i^2 \geq \text{const} > 0$$

for all $\lambda \in \Lambda$, where A_i are the minors of order $(n+m)$ of the matrix

$$\left\| \frac{\partial a}{\partial \lambda}, \frac{\partial b}{\partial \lambda}, \frac{\partial c}{\partial \lambda} \right\|.$$

- 6) $|\cos(\widehat{\tau, n})| \geq \text{const} > 0$ at each point M , where τ and n are, respectively, the unit tangent vector to the curve $x = \text{const}$, $y = \varphi$ and the unit normal to M . Every trajectory $x = \text{const}$, $y = \varphi$ from G intersects M once (if the conditions of the theorem are changed analogously to ⁽¹⁾, then multiple intersections may be allowed).
- 7) In G there are defined k ($k \leq m$) independent integrals of the system $x = \text{const}$, $\dot{y} = Y_0$:

$$\Phi(x, y, t) = c \quad (c = \{c_1, \dots, c_k\}, \Phi = \{\Phi_1, \dots, \Phi_k\}).$$

The values of Φ for $x, y, t \in G$ completely cover some open, possibly unbounded, domain Γ of the variables c_1, \dots, c_k . $\Phi(x, y, t)$ is continuous, has continuous bounded derivatives, $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$ satisfy, with respect to x, t , a Lipschitz condition and admit continuous bounded derivatives with respect to y .

- 8) Let $R(x, y, t)$ be an $(n + k)$ -dimensional vector function defined in G , taking values in the space of the variables $x_1, \dots, x_n, c_1, \dots, c_k$, such that its projections onto the subspaces x and c are respectively X_1 and

$$\frac{\partial \Phi}{\partial x} X_1 + \frac{\partial \Phi}{\partial y} Y_1.$$

Let R have a mean value along every solution $y(t)$ of the degenerate system (under the condition $x, y(t_0), t_0 \in G$):

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} R[x, y(t), t] dt = \bar{R}(x, c), \quad \text{where } c = \Phi[x, y(t), t].$$

The function

$$\bar{R}(x, c) \equiv \bar{R}(r) \quad (r = \{x, c\} \equiv \{x_1, \dots, x_n, c_1, \dots, c_k\})$$

is defined in the domain $r \in \Pi = G_x + \Gamma$ (direct sum), where G_x is the projection of G onto the subspace x . \bar{R} satisfies a Lipschitz condition with respect to x , has continuous bounded derivatives with respect to c , and the function

$$S \equiv R(x, y, t) - \bar{R}[x, \Phi(x, y, t)]$$

is uniformly bounded. The mean value \bar{R} exists uniformly: for every $\delta > 0$ there is a $T(\delta) > 0$ such that, for any value $r = \{x, c\} \in \Pi$ and all solutions of the degenerate system and all t_0 such that $x, y(t_0), t_0 \in G$ and $\Phi[x, y(t), t] = c$,

$$\left| \frac{1}{\tau} \int_{t_0}^{t_0+\tau} \{R[x, y(t), t] - \bar{R}[x, \Phi[x, y(t), t]]\} dt \right| \leq \delta \quad \text{for } \tau \geq T(\delta).$$

- 9) For sufficiently large $T > 0$, for the function

$$N \equiv \int_{t_0}^{t_0+T} S[x, \varphi(x, y_0, t_0, t)] dt$$

the inequalities

$$\left| \frac{\partial N}{\partial y_0} \right|, \quad \left| \frac{\partial N}{\partial t_0} \right| \leq \text{const} < \infty$$

hold.

- 10) For every ε such that $\varepsilon \in (0, \varepsilon_0]$, there exists an open bounded subdomain $G_0(\varepsilon) \subseteq G$ such that some prescribed initial point $l_0(x_0, y_0, t_0) \in G$ lies in all $G_0(\varepsilon)$ together with some fixed neighborhood of radius $\rho > 0$. Every point of $G_0(\varepsilon)$ is separated from M , in the time counted along the integral curve of the system $x = \text{const}, \dot{y} = Y_0$ issuing from it, up to the point of intersection of this curve with M , by an interval Δt such that

$$|\Delta t| \leq K/\varepsilon \quad (K = \text{const} > 0).$$

- 11) For $\varepsilon \in (0, \varepsilon_0]$ there exist subdomains $G_1(\varepsilon) \subset G_0(\varepsilon)$ such that $l_0 \in G_1$ and the distance from the points of G_1 to the boundary of G_0 is not less than $\rho > 0$.
- 12) Let there exist an interval

$$\Delta_1 t(\varepsilon) : [t_0, t_1(\varepsilon)] \quad (t_1 > t_0, t_1 - t_0 \leq K/\varepsilon)$$

such that the integral curve of the basic system $\dot{x} = \varepsilon X, \dot{y} = Y$, issuing from l_0 at $t = t_0$, exists and lies in G for all ε such that $|\varepsilon| \in (0, \varepsilon_0]$ (no assumption of uniqueness of this solution is made).

Denote by $\Delta_2 t(\varepsilon)$ the interval

$$[t_0, t_2(\varepsilon)]$$

such that $t_2 > t_0$, $\Delta_2 t \subseteq \Delta_1 t$, and for $t \in \Delta_2 t$ the integral curve of the system $\dot{x} = \varepsilon X, \dot{y} = Y$ does not leave the

$G_1(\varepsilon)$ (according to the assumptions made, for $\varepsilon \in (0, \varepsilon_0]$ the interval $\Delta_2 t$ exists and, generally speaking, has length $\sim \frac{1}{\varepsilon}$).

Theorem 1. *Under the stated conditions, for arbitrary $K > 0, \delta > 0$, there exists an $\varepsilon_1 > 0$ such that for $0 < \varepsilon \leq \varepsilon_1$ and $t \in \Delta_{2t}(\varepsilon)$ the solution of the averaged system of the first approximation $\dot{\bar{r}} = \varepsilon \bar{R}(\bar{r})$ does not leave $\Pi = G_x + \Gamma$ and $|r - \bar{r}| \leq \delta$, where $\bar{r} = \{\bar{x}(t, \varepsilon), \bar{c}(t, \varepsilon)\}$; \bar{x} and \bar{c} are the solutions of $\dot{\bar{r}} = \varepsilon \bar{R}$, $r = \{x(t, \varepsilon), c(t, \varepsilon)\}$, $x(t, \varepsilon)$ and $y(t, \varepsilon)$ are the solutions of $\dot{x} = \varepsilon X, \dot{y} = Y$, and $c(t, \varepsilon) = \Phi[x(t, \varepsilon), y(t, \varepsilon), t]$, with $x|_{t=t_0} = \bar{x}|_{t=t_0} = x_0, y|_{t=t_0} = y_0, \bar{c}|_{t=t_0} = c|_{t=t_0} = \Phi(x_0, y_0, t_0)$.*

From conditions 1)-12) it follows that in G the system of equations

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} Y_0 = S$$

($u = u(x, y, t) = \{u_1, \dots, u_{n+k}\}$) with the boundary conditions $u|_{x, y, t \in M} = 0$ has a unique continuous solution.

Assume, in addition to the conditions of Theorem 1, that:

- 13) $u(x, y, t)$ has derivatives with respect to x_1, \dots, x_n and is uniformly bounded together with $\partial u / \partial x$.
- 14) X and Y have continuous bounded derivatives with respect to ε .

Theorem 2. *Under the additional conditions 13) and 14), there exists $C > 0$ such that for sufficiently small ε and $t \in \Delta_{2t}(\varepsilon)$, $|r - \bar{r}| \leq C\varepsilon$.*

The equations of higher approximations for the system $\dot{x} = \varepsilon X, \dot{y} = Y$, under the condition that the mean values depend on the choice of solutions of the degenerate system $\dot{x} = \text{const}, \dot{y} = Y_0$, are a generalization of the equations obtained in (2), and are derived in an analogous way. The theorems justifying the higher approximations under such averaging are formulated similarly to Theorems 1 and 2.*

Moscow State University
named after M. V. Lomonosov

Received
7 III 1962

CITED LITERATURE

¹ V. M. Volosov, DAN, 137, No. 1, 21 (1961). ² V. M. Volosov, DAN, 37, No. 5, 1022 (1961). ³ A. N. Tikhonov, A. B. Vasil'eva, V. M. Volosov, *Differential equations containing a small parameter*, Symposium on nonlinear oscillations, September 1961, Kiev, 1961.

* A bibliography on the question considered is available in (1-3).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.