AN EXAMPLE OF THE APPLICATION OF S. L. SOBOLEV' S EMBEDDING THEOREM IN COMPUTATIONAL MATHEMATICS

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Abstract

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In recent years, both in our country and abroad, in questions connected with the solution by the grid method of differential equations with partial derivatives, the well-known embedding theorems of S. L. Sobolev have been applied increasingly widely ([1]). Below, with the aid of embedding theorems, a theoretical justification (proof of convergence) is given for one method which we have already been successfully using for about 4 years for the numerical solution of certain boundary-value problems on high-speed computers.

Suppose it is required to solve approximately the boundary-value problem

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f \quad \text{in } G; \tag{1}$$

$$u = 0 \quad \text{on } S, \tag{2}$$

where G is a finite two-dimensional domain whose boundary S consists of a finite number of line segments parallel to the coordinate axes; moreover, it is assumed that if the domain G is completed to the smallest rectangle R containing the domain G, then from any point $(x, y) \in G$ (respectively, $(x, y) \in R \setminus G$) one can reach the boundary S_0 of the rectangle R by a line segment parallel to one of the coordinate axes and lying entirely in G (respectively, in $R \setminus G$). As to the right-hand side $f(x, y)$, we shall assume that it has continuous first derivatives in G and is equal to zero on S.

If, in programming boundary-value problems for ordinary (one-dimensional) differential equations, the degree of difficulty of programming depends only on the differential equation itself (and the boundary conditions), then the situation is quite different in the case of two-dimensional and three-dimensional boundary-value problems: here the difficulty of programming is determined first of all by the shape of the domain. In programming the method discussed in this paper, however, the question of the shape of the domain is no longer essential, since the matter is reduced to integration over a rectangle (the two-dimensional case) or a parallelepiped (the three-dimensional case). If, in solving problem

(1), (2) by this method, the links (which are parallel to the coordinate axes) of the polygonal boundary γ, where γ is the boundary between G and $R \setminus G$, are continued to their intersection with the boundary S_0 of the rectangle R, then the latter turns out to be represented in a regular manner as a sum of rectangles. This cyclicity in the representation of the given domain, completed to a rectangle, makes it possible conveniently to construct universal programs* for solving boundary-value problems in the case of domains bounded by line segments parallel to the coordinate axes. Moreover, the rectangular domain $R = G \cup G'$ makes it possible, for the solution of the corresponding system of grid equations, very effectively

* A method for constructing universal programs, based on the regular representation of the domain of integration as a sum of rectangles, is set forth in the author' s paper ([2]).

use the fastest-converging iterative method—the method of alternating directions. However, the use of such a method in hand computation is hardly advisable, since calculation of the "fictitious domain" G' requires carrying out idle arithmetic operations, the number of which is the larger, the larger the value of $\operatorname{mes} G' / \operatorname{mes}(G \cup G')$.

In accordance with what has been set forth, instead of problem (1), (2) we shall solve the problem

$$\Delta v = -f \quad \text{in } G; \tag{3}$$

$$\Delta v = 0 \quad \text{in } G'; \tag{4}$$

$$v = 0 \quad \text{on } S_0; \tag{5}$$

$$[v]_{\gamma,G} = [v]_{\gamma,G'}, \qquad \left[\frac{\partial v}{\partial n}\right]_{\gamma,G} = K\left[\frac{\partial v}{\partial n}\right]_{\gamma,G'}, \tag{6}$$

where by γ is meant the "inner boundary" $\bar{\gamma}$ with the corner points removed, at which normals to $\bar{\gamma}$ have no meaning; $K \gg 1$; the notation $[\,]_{\gamma,G}$ (respectively $[\,]_{\gamma,G'}$) means that the quantity enclosed in square brackets is taken on γ from the side of G (respectively G'); n is the direction of the normal to γ from G toward G'. The remaining notation has already been introduced above.

Practice shows that as $K \to \infty$, uniformly in G, $v \to u$, and moreover very rapidly.

Theorem. *Everywhere in G the inequality holds*

$$\max_G |v-u| \leq \frac{C}{\sqrt{K}} \left\{ \iint_G \left[f^2 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right] dx\, dy \right\}^{1/2}, \tag{7}$$

where u (respectively v) is the solution of problem (1), (2) (respectively (3)–(6)), and C is a constant depending only on the form of the domain G.

Proof (brief). If equation (3) is differentiated with respect to x, multiplied by $\partial v/\partial x$, and integrated over G, then after applying Ostrogradskii' s formula we shall have

$$\iint_G \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 \right] dx\, dy - \oint_S \left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial n} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y \partial n} \right) ds =$$

$$= \iint_G \left(\frac{\partial v}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial f}{\partial y} \right) dx\, dy. \tag{8}$$

For the "fictitious domain" G' an analogous relation is valid,

$$\iint_{G'} \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 \right] dx\, dy +$$

$$+ \oint_{S'} \left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial n} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y \partial n} \right) ds = 0, \tag{9}$$

where before the contour integral the sign has been changed to the opposite one, and S' denotes the boundary of the domain G'.

Let us first note that the integrand in the contour integrals (8), (9) outside γ is equal to zero. Indeed, by virtue of (5) and the equality to zero of the function f on the boundary S, for example on the horizontal parts of the boundary S_0, we have

$$\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial n} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y \partial n} = \pm \left[\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial v}{\partial y}\left(\frac{\partial^2 v}{\partial x^2} + f \right) \right] = 0.$$

If we now add to equality (8) equality (9), first multiplied by K, then, after some transformations taking into account the matching conditions (6), we obtain

$$\iint_G \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 v}{\partial x\, \partial y}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 \right] dx\, dy +$$

$$+K\iint_{G'}\left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2+2\left(\frac{\partial^2 v}{\partial x\,\partial y}\right)^2+\left(\frac{\partial^2 v}{\partial y^2}\right)^2\right]dx\,dy=\iint_G\left(\frac{\partial v}{\partial x}\frac{\partial f}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial f}{\partial y}\right)dx\,dy. \tag{10}$$

Applying to the right-hand side of relation (10) the Bunyakovsky-Schwarz inequality and using the inequality estimating the integral of the squares of first derivatives, equal to zero on a part of the boundary, by the integral (over the same domain) of the squares of second derivatives, after repeated strengthening of the inequalities we shall have

$$\iint_{G'}\left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2+\left(\frac{\partial^2 v}{\partial x\,\partial y}\right)^2+\left(\frac{\partial^2 v}{\partial y^2}\right)^2\right]dx\,dy\le\frac{C_1}{K}\iint_G\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right]dx\,dy. \tag{11}$$

Here and below C_i denote constants depending only on the form of the domain.

On the other hand, it is easy to show (even if one requires of the right-hand side f only square integrability over G) the validity of the inequality

$$\iint_{G'}\left[v^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]dx\,dy\le\frac{C_2}{K}\iint_G f^2\,dx\,dy. \tag{12}$$

From (11) and (12) it follows that $v\in W_2^{(2)}(G')$ and

$$\|v\|_{W_2^{(2)}(G')}\le\frac{C_3}{\sqrt{K}}\left[\iint_G\left[f^2+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right]dx\,dy\right]^{1/2},$$

whence, on the basis of the embedding theorem of S. L. Sobolev ((1), p. 91), we immediately conclude that $v\in C(\overline{G'})$ (in particular, v is continuous on $\tilde{\gamma}$) and

$$\|v\|_{C(\tilde{\gamma})}=\max_{\tilde{\gamma}}|v|\le\frac{C_4}{\sqrt{K}}\|f\|_{W_2^{(2)}(G)}. \tag{13}$$

The theorem follows from the fact that the error $v-u$ on the boundary S is equal to v, where it does not exceed the right-hand side of (13) ($\tilde{\gamma}\subset S$), while in G it satisfies the homogeneous equation $\Delta(v-u)=0$ (the maximum principle).

Remark. The theorem admits various generalizations. For example, in the case of the problem

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)-au=-f\quad\text{in } G;\qquad u=0\quad\text{on } S,$$

where the coefficients a_{ij} are continuously differentiable with respect to X in $\overline{G}$ and satisfy the ellipticity condition, $a(X) \geq 0$, $f(X) \in L_2(G)$, instead of (7), on the basis of the second embedding theorem for spaces one can prove the inequality

$$\|v - u\|_{L_p(G)} \leq \frac{C_5}{\sqrt{K}} \|f\|_{L_2(G)},$$

where $p < 2(n-1)/(n-2)$.

From the point of view of constructing universal programs, this theorem is of special interest for the case of equations with discontinuous coefficients (where it is also valid), since in the latter case natural matching conditions are used (along with the artificial conditions (6)).

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CITED LITERATURE

1. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, LSU, 1950.

2. V. K. Saul' yev. *DAN*, **144**, No. 3, 497 (1962).

Note: Figure translations are in progress. See original paper for figures.