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Abstract

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MATHEMATICS

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ON ONE GENERALIZED NOTION OF A SPECTRAL OPERATOR

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The spectral theory of non-self-adjoint operators for which the space decomposes into a direct (discrete or continuous) sum of invariant subspaces was developed with great completeness by N. Dunford and others (^{1}). The decomposition of the space into a direct sum of invariant subspaces is equivalent to the unconditional convergence of expansions in eigenvectors and associated (generalized) vectors. However, verifying that the convergence is unconditional is, as a rule, very difficult. Therefore the existing criteria for the "spectrality" of an operator (^{1}) are not very effective. Moreover, unconditional convergence does not always occur.

In the present paper an attempt is made to construct a spectral theory under the assumption only of completeness of the system of invariant subspaces. Our basic idea is very simple. We immerse the original space $\mathfrak{H}$ into the projective limit $\widehat{\mathfrak{H}}$ of the system of invariant subspaces ($\mathfrak{H}^\Delta$). In the general case, expansions in eigenvectors diverge in the topology of the space $\mathfrak{H}$, but they converge in the topology of the projective limit $\widehat{\mathfrak{H}}$.

1. Generalized decompositions of the identity.

1.1. By a (generalized) decomposition of the identity of a Hilbert space $\mathfrak{H}$ we shall mean an operator-valued set function $P : \Delta \to P(\Delta)$ possessing the following properties: a) the function P is defined on a certain ring $D(P)$ (which, generally speaking, is not a field or a σ-ring) of Borel sets Δ of the complex plane Λ, and the ring $D(P)$ contains every Borel subset of each of its elements; b) the values of the function P are linear bounded operators $P(\Delta)$ mapping the whole space $\mathfrak{H}$ into itself, and: 1) $P(\Delta_1)P(\Delta_2) = P(\Delta_1 \cap \Delta_2)$ for any $\Delta_1, \Delta_2 \in D(P)$; 2) for any decomposition of a set $\Delta \in D(P)$ into disjoint parts $\Delta_1, \Delta_2, ... \in D(P)$ and for any $x, y \in \mathfrak{H}$,

$$\sum_k (P(\Delta_k)x, y) = (P(\Delta)x, y);$$

3) if $P(\Delta)x = 0$ for all $\Delta \in D(P)$, then $x = 0$; 4) if $P^*(\Delta)x = 0$ for all $\Delta \in D(P)$, then $x = 0$, $P^*(\Delta) = [P(\Delta)]^*$.

It is not difficult to construct an example of a generalized decomposition of the identity P that is not equivalent to a decomposition of the identity of a normal operator, i.e. one for which there exists no such continuous automorphism M of the space $\mathfrak{H}$ that all the projectors $M^{-1}P(\Delta)M$, $\Delta \in D(P)$, are self-adjoint operators.

1.2. For an arbitrary set $\Delta \subset \Lambda$ put

$$K_\Delta = \sup\{\|P(\delta)\| : \delta \subset \Delta,\ \delta \in D(P)\}.$$

If $\Delta \in D(P)$, then $K_\Delta < \infty$.

1.3. Denote by $D_0(P)$ the class of all Borel sets $\Delta \subset \Lambda$ for which $K_\Delta < \infty$. For every $\Delta \in D_0(P)$ the sequence of operators $(P(\delta))$ converges in the strong sense when the set δ runs through the upward-directed, with respect to inclusion, system of subsets of the set Δ belonging to the class $D(P)$. Put

$$P(\Delta) = \lim\{P(\delta) : \delta \uparrow \Delta,\ \delta \in D(P)\}.$$

The extended function P is also a decomposition of the identity, and the extension constructed above is maximal (in the sense that P still remains a decomposition of the identity).

1.4. A class of sets $D \subset D(P)$ shall be called **admissible** if the restriction of the function P to the class D is also a resolution of the identity. *The intersection of any two admissible classes is an admissible class.*

1.5. *If D is an admissible class, then the manifold*

$$\bigcup_{\Delta \in D} P(\Delta)\mathfrak{H}$$

is dense in $\mathfrak{H}$.

1.6. *If P is a resolution of the identity defined on the class $D(P)$, then the function*

$$P^* : \Delta \to P^*(\Delta) = [P(\Delta)]^*,$$

defined on the same class of sets $D(P)$, is also a resolution of the identity.

2. Basic and generalized elements

2.1. Let P be a resolution of the identity of the space $\mathfrak{H}$; $D(P)$ the domain of definition of the function P (not necessarily maximal; $D(P) \subset D_0(P)$, see 1.3). For each $\Delta \in D(P)$ put

$$\mathfrak{H}^\Delta = P(\Delta)\mathfrak{H}.$$

The (generalized) sequence $(\mathfrak{H}^\Delta)_{\Delta \in D(P)}$ of Hilbert spaces $\mathfrak{H}^\Delta \subset \mathfrak{H}$ increases with respect to Δ in the sense that

$$P(\Delta_1)\mathfrak{H}^{\Delta_1} \subset \mathfrak{H}^{\Delta_2}, \quad \text{if } \Delta_1 \subset \Delta_2.$$

By the **space of basic elements** $\widetilde{\mathfrak{H}}$ of the Hilbert space $\mathfrak{H}$, corresponding to P and $D(P)$, we shall mean the inductive limit

$$\widetilde{\mathfrak{H}} = \lim\operatorname{ind}_{\Delta}(\mathfrak{H}^{\Delta},\, P(\Delta)).$$

The space $\widetilde{\mathfrak{H}}$ consists of the elements of the union

$$\bigcup_{\Delta\in D(P)} \mathfrak{H}^{\Delta},$$

linear operations in $\widetilde{\mathfrak{H}}$ are induced from $\mathfrak{H}^{\Delta}$, $\Delta \in D(P)$, and the defining system of neighborhoods of zero in $\widetilde{\mathfrak{H}}$ is formed by sets of the form

$$\widetilde{U} = \bigcup_{\Delta\in D(P)} U^{\Delta},$$

where U^{Δ} is a neighborhood of zero in $\mathfrak{H}^{\Delta}$.

2.2. Note that the (generalized) sequence $(\mathfrak{H}^{\Delta})_{\Delta\in D(P)}$ of Hilbert spaces $\mathfrak{H}^{\Delta} \subset \mathfrak{H}$ decreases with respect to Δ in the sense that

$$P(\Delta_1)\mathfrak{H}^{\Delta_2} \subset \mathfrak{H}^{\Delta_1}, \quad \text{if } \Delta_1 \subset \Delta_2.$$

By the **space of generalized elements** $\widehat{\mathfrak{H}}$ of the Hilbert space $\mathfrak{H}$, corresponding to P and $D(P)$, we shall mean the projective limit

$$\widehat{\mathfrak{H}} = \lim\operatorname{pr}_{\Delta}(\mathfrak{H}^{\Delta};\, P(\Delta)).$$

The space $\widehat{\mathfrak{H}}$ consists of (generalized) sequences

$$\hat{x} = (x_{\Delta})_{\Delta\in D(P)}$$

such that $x_{\Delta} \in \mathfrak{H}^{\Delta}$ and

$$P(\Delta_1)x_{\Delta_2} = x_{\Delta_1}, \quad \text{if } \Delta_1 \subset \Delta_2.$$

By definition

$$(x_{\Delta}) + (y_{\Delta}) = (x_{\Delta} + y_{\Delta})$$

and

$$\alpha(x_{\Delta}) = (\alpha x_{\Delta}), \quad \alpha \in \Lambda.$$

The defining system of neighborhoods of zero in $\widehat{\mathfrak{H}}$ is formed by sets of the form

$$\widehat{U} = \{\hat{x} : \hat{x} \in \widehat{\mathfrak{H}},\, P(\Delta_0)\hat{x} \in U^{\Delta_0}\},$$

where U^{Δ_0} is a neighborhood of zero in $\mathfrak{H}^{\Delta_0}$ (depending on $\widehat{U}$), and

$$P(\Delta_0)(x_{\Delta}) = x_{\Delta_0}; \quad \Delta, \Delta_0 \in D(P).$$

2.3. Let $\widetilde{\mathfrak{H}}_*$ be the space of basic elements of the space $\mathfrak{H}$, corresponding to the resolution of the identity P^*, defined on the class $D(P^*) = D(P)$ (see 1.6). *The space of generalized elements $\widehat{\mathfrak{H}}$ is the space of linear continuous functionals on $\widetilde{\mathfrak{H}}_*$.* Namely, the formula

$$\langle \tilde{x}_*, \hat{x} \rangle = (\tilde{x}_*, P(\Delta)\hat{x})$$

for all $\tilde{x}_* \in \widetilde{\mathfrak{H}}_*$, $\Delta \in D(P)$, establishes a one-to-one correspondence between functionals $\tilde{x}_* \in \widetilde{\mathfrak{H}}'_*$ and generalized elements $\hat{x} \in \widehat{\mathfrak{H}}$ (recall that $P(\Delta_0)(x_\Delta) = x_{\Delta_0}$; $\langle\, ,\, \rangle$ denotes the duality of the spaces $\widetilde{\mathfrak{H}}_*$ and $\widetilde{\mathfrak{H}}'_*$, and $(\, ,\,)$ is the scalar product in $\mathfrak{H}$).

2.4. For each $x \in \mathfrak{H}$ there exists a unique element $\hat{x} \in \widehat{\mathfrak{H}}$ such that

$$P(\Delta)\hat{x} = P(\Delta)x$$

for all $\Delta \in D(P)$. In this sense $\mathfrak{H} \subset \widehat{\mathfrak{H}}$, and since, moreover, $\widetilde{\mathfrak{H}} \subset \mathfrak{H}$, we have

$$\widetilde{\mathfrak{H}} \subset \mathfrak{H} \subset \widehat{\mathfrak{H}}.$$

The manifold $\widetilde{\mathfrak{H}}$ is dense in the topological space $\widehat{\mathfrak{H}}$.

2.5. For each $\Delta \in D(P)$, by $\widetilde{P}(\Delta)$ ($\widehat{P}(\Delta)$) we denote the restriction (extension) of the operator $P(\Delta)$ to the manifold $\widetilde{\mathfrak{H}}$ ($\widehat{\mathfrak{H}}$: $\widehat{P}(\Delta)\hat{x} = JP(\Delta)x$, where J is the operator of embedding of $\mathfrak{H}$ into $\widehat{\mathfrak{H}}$; this extension is at the same time an extension by continuity). For every $x \in \widetilde{\mathfrak{H}}$ ($\hat{x} \in \widehat{\mathfrak{H}}$) we have

$$\tilde{x} = \int \widetilde{P}(d\lambda)\tilde{x} = \lim_{\Delta\nearrow} \widetilde{P}(\Delta)\tilde{x} \quad \left(\hat{x} = \int \widehat{P}(d\lambda)\hat{x} = \lim_{\Delta\nearrow} \widehat{P}(\Delta)\hat{x}\right)$$

($\Delta \nearrow$ means that Δ runs through the upward-directed, with respect to inclusion $\subset$, system of mno-

$D(P)$). Consequently, for every $x \in \mathfrak{H}$ we have the expansion $x = \int P(d\lambda)x$, where the integral exists in the sense of the topology of the projective limit $\mathfrak{H}$.

3. Operators permutable with the resolution of the identity

3.1. A closed linear operator A with domain of definition $\mathfrak{D}(A)$, dense in the space $\mathfrak{H}$, will be called *permutable* with the resolution of the identity P if: α) $AP(\Delta) \supset P(\Delta)A$ for every $\Delta \in D(P)$; β) the class D_A of all those sets $\Delta \in D(P)$ for which $P(\Delta)\mathfrak{H} \subset \mathfrak{D}(A)$ is an admissible class (see 1.4). If the operator A is permutable with the resolution of the identity P, then, as is not difficult to see, the operator $AP(\Delta)$ is bounded for every $\Delta \in D(P)$. The totality of all operators permutable with P will be denoted by $\mathfrak{A}(P)$.

3.2. Let $\Delta \to \widetilde{A}(\Delta)$ be an operator-valued set function defined on some admissible class of sets D, and suppose that for each $\Delta \in D$, $\widetilde{A}(\Delta)$ is a bounded linear operator $\mathfrak{H} = \mathfrak{D}(A(\Delta)) \to \mathfrak{H}$. In order that there exist an operator $A \in \mathfrak{A}(P)$ for which $AP(\Delta) = \widetilde{A}(\Delta)$, $\Delta \in D$, it is necessary and sufficient that the condition

$$P(\Delta_1)\widetilde{A}(\Delta_2) = \widetilde{A}(\Delta_2)P(\Delta_1) = A(\Delta_1 \cap \Delta_2) \quad \text{for all } \Delta_1, \Delta_2 \in D \tag{a}$$

be satisfied.

3.3. Let $\Delta \to \widetilde{A}_i(\Delta)$ be functions satisfying the conditions of 3.2, and let $A_i \in \mathfrak{A}(P)$ be the operator determined by this function, $A_{iP}(\Delta) = \widetilde{A}_i(\Delta)$, $i = 1, 2$. If $\widetilde{A}_1(\Delta) = \widetilde{A}_2(\Delta)$ for all $\Delta \in D_1 \cap D_2$, then $A_1 = A_2$; in particular $\mathfrak{D}(A_1) = \mathfrak{D}(A_2)$.

3.4. If $A \in \mathfrak{A}(P)$, then $A^* \in \mathfrak{A}(P^*)$ and $D_A = D_{A^*}$.

3.5. Denote by $\widetilde{\mathfrak{A}}(P)$ ($\widehat{\mathfrak{A}}(P)$) the class of all linear continuous operators $\mathfrak{H} \to \widetilde{\mathfrak{H}}$ ($\widetilde{\mathfrak{H}} \to \mathfrak{H}$) that are permutable with each of the operators $\widetilde{P}(\Delta)$ ($\widehat{P}(\Delta)$), $\Delta \in D(P)$. In the following propositions a description is given of the relations existing among the elements of the classes $\widetilde{\mathfrak{A}}(P)$, $\mathfrak{A}(P)$, and $\widehat{\mathfrak{A}}(P)$.

1°. Let $\widetilde{A} \in \widetilde{\mathfrak{A}}(P)$. The operator $\widetilde{A}$, considered in the space $\mathfrak{H} \supset \widetilde{\mathfrak{H}}$, admits a closure A; moreover $A \in \mathfrak{A}(P)$.

2°. Let $A \in \mathfrak{A}(P)$. Since the restriction of the function P to the class D_A (see 3.1β) is also a resolution of the identity, we shall assume that $D(P) = D_A$. Under this assumption $\widetilde{\mathfrak{H}} \subset \mathfrak{D}(A)$. If $\widetilde{A}$ is the restriction of the operator A to $\widetilde{\mathfrak{H}}$, then $\widetilde{A} \in \widetilde{\mathfrak{A}}(P)$.

3°. Let $\widetilde{A} \in \widetilde{\mathfrak{A}}(P)$. The operator $\widetilde{A}$ is continuous in the sense of the topology induced in $\widetilde{\mathfrak{H}}$ by the topology of the space $\mathfrak{H} \supset \widetilde{\mathfrak{H}}$, and since $\widetilde{\mathfrak{H}}$ is dense in $\widehat{\mathfrak{H}}$, $\widetilde{A}$ extends by continuity to the space $\widehat{\mathfrak{H}}$. Denoting this extension by $\widehat{A}$, we have $\widehat{A} \in \widehat{\mathfrak{A}}(P)$.

4°. Let $\widehat{A} \in \widehat{\mathfrak{A}}(P)$. Put

$$\mathfrak{D}(A) = \{\hat{x} : \hat{x} \in \mathfrak{H},\ \widehat{A}\hat{x} \in \mathfrak{H}\}, \qquad A \subset \widehat{A}.$$

Then $A \in \mathfrak{A}(P)$.

3.6. The set $\mathfrak{A}(P)$ is transformed into an algebra over the field of complex numbers if the algebraic operations on the operators $A \in \mathfrak{A}(P)$ are defined by means of the corresponding operations on the (bounded) operators $\widetilde{A}(\Delta) = AP(\Delta)$ (see 3.2 and condition (a)). The sets $\widetilde{\mathfrak{A}}(P)$ and $\widehat{\mathfrak{A}}(P)$ consist of operators defined on the whole space ($\widetilde{\mathfrak{H}}$ or $\widehat{\mathfrak{H}}$), and therefore are algebras with respect to the usual operations on operators. The one-to-one correspondences $A \to \widetilde{A}$ and $A \to \widehat{A}$ (see 3.5) are isomorphisms of the algebras $\mathfrak{A}(P)$, $\widetilde{\mathfrak{A}}(P)$, and $\widehat{\mathfrak{A}}(P)$. These correspondences are also continuous mappings if convergence of a sequence $(A_\alpha) \in \mathfrak{A}(P)$ is defined as convergence (in norm in $\mathfrak{H}$) of the sequences $(A_\alpha P(\Delta))$ for every $\Delta \in D(P)$.

3.7. Proposition 3.2 can naturally be used for constructing analytic functions of elements of the algebra $\mathfrak{A}(P)$. For every $\Delta \in D(P)$

we first construct the operator $f(AP(\Delta))$, where f is a function analytic in some neighborhood of the spectrum of the operator $A \in \mathfrak{A}(P)$; we verify that the function $\Delta \to f(A(\Delta))$ satisfies condition (3.1), and therefore defines an operator $f(A) \in \mathfrak{A}(P)$.

4. Spectral operators. A linear operator $A : \mathfrak{D}(A) \to \mathfrak{H}$, $\overline{\mathfrak{D}(A)} = \mathfrak{H}$, will be called **spectral** if there exists a resolution of the identity P such that $A \in \mathfrak{A}(P)$ and, for every $\Delta \in D_A$, the spectrum of the operator A, considered on the invariant subspace $P(\Delta)\mathfrak{H}$, is contained in the closure $\overline{\Delta}$ of the set Δ. An operator $A \in \mathfrak{A}(P)$ is spectral if and only if

$$A = \int \lambda P(d\lambda) + N,$$

where the integral converges in the topology of $\mathfrak{A}(P)$, while $N \in \mathfrak{A}(P)$ and $NP(\Delta)$ is a quasinilpotent operator for every $\Delta \in D_A$.

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REFERENCES

1. N. Dunford, *Sborn. per. Matematika*, **4**, 1, 1960.

Note: Figure translations are in progress. See original paper for figures.

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