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Abstract

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MATHEMATICS

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ON REDUCING A GAME WITH PERFECT RECALL TO A MATRIX GAME

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1. In this paper we consider the problem of solving finite zero-sum positional games of two persons with perfect recall (for notions and notation not explained, see (^{1})). It is known that any finite positional game can be normalized, and our task is to make use of the advantages of games with perfect recall that follow from Kuhn' s theorem on behavior strategies (^{2}).

We shall reduce the problem of finding optimal behavior strategies to the solution of a matrix game with constraints which, as shown in (^{4}), is effectively reducible to a problem of linear programming. Apparently, analogous reduction methods can also be used in finding signaling and decomposed strategies (^{3}, ^{5}).

2. To define our matrix game, we shall construct auxiliary graphs– "information trees" of the first and second player. In doing so, we shall use the partial ordering, existing under perfect recall, of the set consisting of the player' s information sets and of the alternatives in these information sets.

We shall call an **information tree** T_1 of the first player the tree formed according to the following rules:

1) To each information set $U_k^{(1)}$ of player I in the original game there corresponds one (its own) vertex u_k of the information tree. We shall call such vertices **basic**.
2) To each alternative of each information set $U_k^{(1)} : v$ of player I in the original game there corresponds one (its own) vertex $v_{k,v}$ of the information tree. We shall call such vertices auxiliary.
3) If the initial position 0 of the original game tree does not belong to I_1, then one more auxiliary vertex v_0, corresponding to it and preceding all the other vertices, is introduced.

4) Each basic vertex u_k is followed immediately by auxiliary vertices $v_{k,v}$, whose number t_k is equal to the number of alternatives of the information set $U_k^{(1)}$.

5) The number of vertices immediately following an auxiliary vertex $v_{k,v}$ is determined as follows: a) if there are no information sets $U_l^{(1)}$ following $U_k^{(1)} : v$, then the vertex $v_{k,v}$ is terminal; b) the vertex $v_{k,v}$ is immediately followed by as many basic vertices as there are pairwise incomparable information sets of player I following $U_k^{(1)} : v$ and preceding all the other information sets of player I that follow $U_k^{(1)} : v$. In addition, if there exists a terminal position W following $U_k^{(1)} : v$ and such that the path from $U_k^{(1)} : v$ to W does not intersect player I' s information sets, then the vertex $v_{k,v}$ is immediately followed by one more auxiliary vertex, which is terminal.

6) The player' s information tree preserves the partial ordering of this player' s information sets and of the alternatives in them.

Similarly, the information tree of player II, T_2, is defined.

Each terminal position $\mathfrak{w}_i$ of the tree T_i (the set of which we shall denote by $\mathfrak{w}_i$) corresponds to a certain set of terminal positions W of the original tree for which $\mathfrak{P}_i(W)$ coincide. We shall denote this set by $W(\mathfrak{w}_i)$. Note that, whatever $\mathfrak{w}_1 \in \mathfrak{w}_1$ and $\mathfrak{w}_2 \in \mathfrak{w}_2$ may be, the set $W(\mathfrak{w}_1) \cap W(\mathfrak{w}_2)$ contains no more than one element.

3. Let a nonnegative number $p_x \geq 0$ be assigned to each vertex x of the tree T_1, and suppose that the following conditions are satisfied: a)

$$p_{u_k} = \sum_{\nu=1}^{t_k} p_{v_k,\nu},$$

for all basic vertices u_k; b) for any auxiliary vertex v, $p_v = p_x$ for all $x \in f_{T_1}^{-1}(v)$; c) for the initial position x_0, $p_{x_0} = 1$.

It is easy to see that every such set can be reconstructed from the numbers $p_{\mathfrak{w}_1}$ corresponding to the terminal vertices $\mathfrak{w}_1$. A set $\pi_1 = \{p_{\mathfrak{w}_1}\}$, $\mathfrak{w}_1 \in \mathfrak{w}_1$, from which the numbers p_x can be reconstructed for all vertices x of the tree T_1, will be called a **quasistrategy** of player I. The set of quasistrategies Π_1 of player I will be a bounded convex polyhedron. The quasistrategy $\pi_2 = \{q_{\mathfrak{w}_2}\}$, $\mathfrak{w}_2 \in \mathfrak{w}_2$, of player II is defined analogously.

We shall call a **quasigame** Γ the problem of finding the minimax of the bilinear form

$$\pi_1 K \pi_2 = \sum_{\mathfrak{w}_1 \in \mathfrak{w}_1} \sum_{\mathfrak{w}_2 \in \mathfrak{w}_2} K(\mathfrak{w}_1, \mathfrak{w}_2) p_{\mathfrak{w}_1} q_{\mathfrak{w}_2}$$

Fig. 1. Game tree

Figure 1: Fig. 1. Game tree

(where $K(\mathfrak{w}_1, \mathfrak{w}_2) = h_1(W)$ if $W \in W(\mathfrak{w}_1) \cap W(\mathfrak{w}_2)$, and $K(\mathfrak{w}_1, \mathfrak{w}_2) = 0$ if $W(\mathfrak{w}_1) \cap W(\mathfrak{w}_2) = \Lambda$) for $\pi_1 \in \Pi_1$ and $\pi_2 \in \Pi_2$.

It is known (^{4,5}) that

$$\min_{\Pi_2} \max_{\Pi_1} \pi_1 K \pi_2 = \max_{\Pi_1} \min_{\Pi_2} \pi_1 K \pi_2$$

and that the problem of finding optimal quasistrategies π_1 and π_2 is equivalent to a linear-programming problem.

4. Finding optimal quasistrategies in the quasigame turns out to be equivalent to finding optimal behavior strategies in the original game, as follows from the following theorems.

Theorem 1. To each quasistrategy π_i of player i there corresponds, in a one-to-one way, his behavior strategy in the original positional game $\mu_i(\pi_i)$.

Theorem 2. The value of the bilinear form $\pi_1 K \pi_2$ is equal to the payoff in the original positional game when the behavior strategies $\mu_1(\pi_1)$ and $\mu_2(\pi_2)$ are used.

5. In conclusion let us consider an example. The tree of the positional game is shown in Fig. 1. Player I has four information sets, player II three. The 17 terminal positions have been renumbered. The payoff to player I in position k is equal to h_k.

Fig. 1. Game tree

The information trees of the players are shown in Figs. 2 and 3. The basic positions are circled. The terminal positions of T_1 are denoted by Latin letters, and those of T_2 by Greek letters.

Here the quasistrategy $\pi_1 = (p_a, p_b, \dots, p_g)$ of player I satisfies the conditions

$$p_a + p_b = p_c, \qquad p_d + p_e = p_f + p_g, \qquad p_c + p_d + p_e = 1,$$

quasi-strategy $\pi_2 = (q_\alpha, q_\beta, \dots, q_\eta)$ of player II—the conditions

$$q_\alpha + q_\beta = q_\gamma + q_\delta, \qquad q_\alpha + q_\beta + q_\varepsilon + q_\zeta + q_\eta = 1.$$

The matrix K and the sets $W(w_1)$ and $W(w_2)$ are as follows:

Player I	$W(w_1)$	Player II / $W(w_2)$	α	β	γ	δ	ε	ζ	η
w_1	$W(w_1)$	w_2	1, 3	2, 4	8, 10	9, 11	5, 12, 13	6, 14, 15	7, 16, 17
a	1, 2		h_1	h_2	0	0	0	0	0
b	3, 4		h_3	h_4	0	0	0	0	0
c	5, 6, 7		0	0	0	0	h_5	h_6	h_7
d	8, 9, 12		0	0	h_8	h_9	h_{12}	0	0
e	10, 11, 13		0	0	h_{10}	h_{11}	h_{13}	0	0
f	14, 16		0	0	0	0	0	h_{14}	h_{16}
g	15, 17		0	0	0	0	0	h_{15}	h_{17}

6. Remark 1. The same reduction method is also applicable to games with an intermediary, as well as to games in which only one player has memory.

Fig. 2. Information tree of player I

Fig. 3. Information tree of player II

Remark 2. The use of the proposed method does not preclude other techniques for reducing the dimensions of the game. Thus, in the example considered one can separately determine the optimal behavior of the players on the sets $U_2^{(1)}$ and $U_2^{(2)}$, by solving a 2×2 matrix game.

Remark 3. In the case where no more than one information set immediately follows each alternative of each information set of player i, one can construct such a quasi-game in which Π_i coincides with the simplex of probability vectors. Such games reduce especially simply to a linear programming problem.

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Note: Figure translations are in progress. See original paper for figures.