# MATHEMATICS

V. PONOMAREV

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## Abstract

## Full Text

**MATHEMATICS**

**V. PONOMAREV**

# ON PARACOMPACT SPACES AND THEIR CONTINUOUS MAPPINGS

*(Presented by Academician P. S. Aleksandrov, 23 XI 1961)*

§ 1. In this note it is first of all established (Theorem 1 and the basic definition in § 2) that all paracompacts, and only they, admit a sufficiently good approximation by projection spectra (in the classical sense established by P. S. Aleksandrov ($^{1a}$) and generalized by A. G. Kurosh ($^{1b}$)). From this theorem it is then deduced (Theorem 2) that all paracompacts, and only they, are images of perfectly zero-dimensional spaces* under perfect** continuous irreducible*** mappings. Moreover, for each paracompact $X$ the perfectly zero-dimensional space $\dot{X}$, whose image is the paracompact $X$, is determined by a certain natural one-to-one construction; therefore I call this space $\dot{X}$ an absolute preimage, or simply the absolute, of the space $X$. It turns out (Theorem 5) that two paracompacts can be perfectly and irreducibly (but, generally speaking, many-valuedly) mapped onto one another if and only if they have one and the same absolute; moreover, every one-to-one irreducible perfect mapping $f : X \to Y$ determines a certain homeomorphism $\dot{f} : \dot{X} \to \dot{Y}$ between the absolutes $\dot{X}$ and $\dot{Y}$, and in turn is determined by it (Theorem 4).

§ 2. A projection spectrum is a directed set $S = \{\alpha, \mathfrak{d}_{\alpha}^{\alpha'}\}$ of simplicial complexes $\alpha$ with simplicial mappings (projections) $\mathfrak{d}_{\alpha}^{\alpha'}$ of the complex $\alpha'$ onto $\alpha$ (for all $\alpha' > \alpha$). In addition the transitivity condition is fulfilled: if $\alpha'' > \alpha' > \alpha$, then $\mathfrak{d}_{\alpha}^{\alpha''} = \mathfrak{d}_{\alpha}^{\alpha'} \mathfrak{d}_{\alpha'}^{\alpha''}$. A thread of the spectrum is a system $\xi = \{t_\alpha\}$ of simplices, one $t_\alpha$ from each $\alpha$, having the property that for $\alpha' > \alpha$ one always has $\mathfrak{d}_{\alpha}^{\alpha'} t_{\alpha'} = t_\alpha$. A thread $\xi' = \{t'_\alpha\}$ embraces the thread $\xi = \{t_\alpha\}$ if for every $\alpha$ we have $t_\alpha \geq t'_\alpha$ (this means that $t_\alpha$ is a proper or improper face of the simplex $t'_\alpha$). A thread $\xi$ is called maximal if it has no thread embracing it that is distinct from it. The space $\widetilde{S} = \lim S$ of the spectrum $S$ is the set of all maximal threads. The topology on this set is introduced by means of a base consisting of elementary open sets $Ot_\alpha$; here $Ot_\alpha$

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* A space $\dot{X}$ is called perfectly zero-dimensional if into every open covering $\omega$ of the space $\dot{X}$ one can inscribe a covering $\omega'$ whose elements are pairwise disjoint (open-and-closed) sets.

** A one-to-one mapping $f : X \to Y$ is called perfect if it is closed, continuous, and the preimages $f^{-1}y$ of all points $y \in Y$ are bicompact. A many-valued mapping is perfect ($^2$) if, in addition, the images $fx$ of all points $x \in X$ are bicompact. For a many-valued perfect mapping $f : X \to Y$ there exists a space $Z$ and one-to-one mappings $f_X : Z \to X$ and $f_Y : Z \to Y$ such that $f = f_Y f_X^{-1}$.

*** A one-to-one mapping $f$ of a space $X$ onto $Y$ is called irreducible if, whatever closed subset $A \subset X,\ A \neq X$, we always have $fA \neq Y$. A many-valued perfect mapping $f : X \to Y$ is called irreducible if a space $Z$ and irreducible one-to-one perfect mappings $f_X : Z \to X,\ f_Y : Z \to Y$ can be found in such a way that $f = f_Y f_X^{-1}$.

is the set of all maximal threads $\xi'$ for which, for a given fixed $\alpha$, we have $t_\alpha \geq t'_\alpha,\ t'_\alpha \in \xi'$. The space $\widetilde{S}$ with this topology is always a $T_1$-space. Let $e_\alpha$ be an arbitrary vertex of the given fixed complex $\alpha \in S$. By $\Phi_{e_\alpha}$ we denote the set of all maximal threads $\xi$ for which (for the given $\alpha$) we have $e_\alpha \leq t_\alpha \in \xi$. The sets $\Phi_{e_\alpha}$ are closed. For a given fixed $\alpha$, the sets $\Phi_{e_\alpha}$, where $e_\alpha$ runs through the totality of all vertices of the complex $\alpha$, form a closed covering $\varphi_\alpha$ of the space $\widetilde{S}$.

**Remark.** The nerve of the covering $\varphi_\alpha$ is always a subcomplex of the complex $\alpha$. If the spectrum $S$ is complete*, then the nerve of the covering $\varphi_\alpha$ coincides with the complex $\alpha$.

**Definition.** A spectrum $S$ is called **regular** if, whatever the point $\xi = \{t_\alpha\}$ and the given $\alpha$ may be, there exists an $\alpha' > \alpha$ such that, for $t_{\alpha'} \in \xi$, $t_{\alpha'} = |e^0_{\alpha'}, \dots, e^r_{\alpha'}|$, we have

$$\bigcup_{i=1}^{r} \Phi_{e^i_{\alpha'}} \subseteq Ot_\alpha.$$

The space of a regular spectrum is always a regular space.

**Basic definition.** A spectrum $S$ is called **uniform** if, whatever the covering $\omega = \{U\}$ of its space $S$ by elementary open sets may be, there exists an $\alpha \in S$ such that the covering $\varphi_\alpha$ is inscribed in the covering $\omega$.

**Theorem 1.** *The space of every regular uniform spectrum is paracompact; conversely, every paracompact space is the space of a certain complete regular uniform spectrum.*

The proof of the first assertion rests on the fact that every covering $\varphi_\alpha$ is locally finite.

For the proof of the second assertion, in the paracompact $X$ one takes the directed (in the natural way) system of all decompositions**. Then the nerves of these decompositions, with the natural projections, form the required spectrum $S$. This spectrum $S$ is called the** spectrum of the paracompact**

$X$. It is proved that the spectrum $S$ is uniform, regular, and complete, and that its space is homeomorphic to the paracompact $X$.

§ 3. Let $S$ be the spectrum of a paracompact $X$.

**Proposition A.** *Each thread* $\xi' = \{t'_\alpha\}$ *of the spectrum* $S$ *is contained in a unique maximal thread* $\xi = \{t_\alpha\}$.

**Proposition B.** *Each thread* $\xi = \{t_\alpha\}$ *of the spectrum* $S$ *envelops some thread* $\xi_0 = \{e_\alpha\}$, *where* $e_\alpha$ *is a vertex of the simplex* $t_\alpha$.

Further, for any $\alpha \in S$, by $\dot{\alpha}$ we denote the zero-dimensional complex consisting of all vertices of the complex $\alpha$. Keeping the projections $\mathfrak{D}_\alpha^{\alpha'}$ of the spectrum $S$ (but considering them as mappings of the complex $\dot{\alpha}'$ onto $\dot{\alpha}$), we obtain a spectrum $\dot{S} = \{\dot{\alpha}, \mathfrak{D}_\alpha^{\alpha'}\}$, uniquely determined by the spectrum $S$ (and, consequently, by the paracompact $X$), called the **complete relaxation of the spectrum** $S$. Its space (the uniquely determined paracompact $X$) we denote by $\dot{X}$: $\dot{X} = \lim \dot{S}$, and call it the **absolute** of the space $X$. The elementary open and elementary closed sets of the spectrum $\dot{S}$ coincide—these are the sets (open-and-closed)

$$\tilde{e}_{\alpha_0} = E(\xi' = \{\dot{e}_\alpha\},\ \dot{e}_{\alpha_0} = e_{\alpha_0}).$$

Each thread $\dot{x} = \{e_\alpha\}$ of the spectrum $\dot{S}$ is maximal; at the same time it is a thread of the spectrum $S$.

Assigning to each thread $\dot{x} = \{e_\alpha\} \in \dot{X}$ the unique (by Proposition A) maximal thread $x$ of the spectrum $S$ that contains it, we obtain a mapping $\pi_X : \dot{X} \to X$, which is (by Proposition B) a mapping of the space $\dot{X}$ onto the space $X$.

**Theorem 2′.** The mapping $\pi_X$ just constructed (cf. (3)) of the absolute $\dot{X}$ of the space $X$ onto the space $X$ is a perfect irreducible mapping.

**Theorem 2″.** The absolute $\dot{X}$ of every paracompactum $X$ is a perfectly zero-dimensional (and hence strongly paracompact) space, for which

$$\dim \dot{X} = \operatorname{ind} \dot{X} = \operatorname{Ind} \dot{X} = 0.$$

Since, on the other hand, the image of every paracompactum under a perfect mapping is always a paracompactum, the following characterization of paracompact spaces follows from Theorems 2′ and 2″:

**Theorem 2.** Among regular spaces, the paracompacta, and only they, are images of perfectly zero-dimensional spaces under perfect irreducible mappings.

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* The spectrum $S = \{\alpha, \mathfrak{D}_\alpha^{\alpha'}\}$ is called complete if, for every simplex $t_\alpha \in \alpha$, there exists a maximal thread $\xi \ni t'_\alpha \geq t_\alpha$.

** A decomposition is a locally finite covering whose elements are the closures of pairwise intersecting open sets.

§ 4. In this and the following paragraphs it will be convenient for us to denote by $\alpha$ an arbitrary covering of the paracompactum $X$, by $|\alpha|$ its nerve, and by $\dot\alpha$ the set of all vertices of the nerve $|\alpha|$.

Let $f$ be an irreducible one-valued perfect mapping of a paracompactum $X$ onto a paracompactum $Y$.

**Fundamental lemma.** Under the mapping $f$, every covering $\alpha = \{A_\lambda\}$ of the space $X$ is mapped onto a covering $\beta_\alpha = f\alpha = \{fA_\lambda\}$ of the space $Y$ in a one-to-one manner (in the sense that the correspondence $A_\lambda^\alpha \leftrightarrow fA_\lambda^\alpha$ between the elements of the coverings $\alpha$ and $\beta_\alpha$ is one-to-one). Moreover, for every covering $\beta$ of the space $Y$ one can find at least one covering $\alpha$ of the space $X$ such that $f\alpha = \beta$.

From this lemma it follows:

**Theorem 3** (spectral definition of a mapping). An irreducible perfect mapping $f$ of a paracompactum $X$ onto a paracompactum $Y$ determines a passage from the spectrum $S_X = \{\alpha, \mathfrak{W}_\alpha^{\alpha'}\}$ of the paracompactum $X$ to the spectrum $S_{XY} = \{\beta_\alpha, \mathfrak{W}_{\beta_\alpha}^{\beta_{\alpha'}}\}$, having as its limit the paracompactum $Y$, in which the $\beta_\alpha$ are directed by the indices $\alpha$; each complex $|\beta_\alpha|$ has the same set of vertices $\dot\beta_\alpha = \dot\alpha$ as $\alpha$, and contains $\dot\alpha$ as a subcomplex; the mapping $f$ also specifies those new simplexes in $\alpha$ by which $|\alpha|$ must be supplemented in order to obtain $|\beta_\alpha|$. The projection $\mathfrak{W}_{\beta_\alpha}^{\beta_{\alpha'}}$ is a simplicial mapping of the complex $|\beta_{\alpha'}|$ onto $|\beta_\alpha|$, given on $|\alpha'| = \dot\alpha'$ as the mapping $\mathfrak{W}_\alpha^{\alpha'}$ (and hence is the unique extension of the mapping $\mathfrak{W}_\alpha^{\alpha'} : |\alpha'| \to |\alpha|$ to a simplicial mapping $|\beta_{\alpha'}| \to |\beta_\alpha|$). The mapping $f$ itself is constructed as follows: each maximal thread $x$ of the spectrum $S_X$, being a thread of the spectrum $S_{X,Y}$, is contained in a unique maximal thread $y = fx$ of the spectrum $S_{X,Y}$. From the spectrum $S_Y$ of the paracompactum $Y$, the spectrum $S_{X,Y}$ is obtained by multiplication (see, for example, (5), p. 37) and subsequent weakening of the order.

§ 5. Since $f$ induces a one-to-one mapping of $\dot\alpha$ onto $\dot\beta_\alpha$ and carries the projection $\mathfrak{W}_\alpha^{\alpha'}$ into $\mathfrak{W}_{\beta_\alpha}^{\beta_{\alpha'}}$, $f$ induces a homeomorphism $\dot f$ of the absolute $\dot X$ onto the absolute $\dot Y$. The mapping $\pi_Y : \dot Y \to Y$ can be interpreted as the passage from a thread $\dot y = \dot f \dot x$ of the spectrum $\dot S_Y$ to the thread $\dot x = \dot f^{-1} \dot y$, then to the unique maximal thread $x = \pi_X \dot x$ of the spectrum $S_X$ containing this thread, and finally to the maximal thread $fx$ of the spectrum $S_{X,Y}$ containing this latter thread. In other words:

$$\pi_Y = f \pi_X \dot f^{-1}$$

and, therefore,

$$f = \pi_Y \dot f \pi_X^{-1}. \tag{1}$$

Conversely, if a single-valued mapping is representable in the form (1), then it is perfect and irreducible. Thus:

**Theorem 4.** In order that there exist a single-valued irreducible perfect mapping $f$ of a paracompactum $X$ onto a paracompactum $Y$, it is necessary and sufficient that the absolutes $\dot{X}$ and $\dot{Y}$ be homeomorphic and that there be a single-valued mapping $f : X \to Y$, where $\dot{f} : \dot{X} \to \dot{Y}$ is some homeomorphism between $\dot{X}$ and $\dot{Y}$.

If this condition is fulfilled, then every irreducible perfect mapping $f : X \to Y$ is determined by some homeomorphism $\dot{f} : \dot{X} \to \dot{Y}$ between $\dot{X}$ and $\dot{Y}$ by formula (1).

Formula (1) may be interpreted as follows: taking $\dot{X} = \dot{Y}$, we may regard $X$ as the quotient space $I = \{\pi_X^{-1}x\}$, and $Y$ as the quotient space $II = \{\pi_Y^{-1}y\}$ of the same space $\dot{X} = \dot{Y}$. By means of a certain topological mapping of the space $\dot{X}$ onto itself we can arrange that each element of the quotient space $I$ be contained in some element of the quotient space $II$. Assigning to each element of the first quotient space the element of the second that contains it, we obtain the mapping $f$.

In conclusion we formulate an easily proved proposition:

**Theorem 5.** In order that two paracompacta could be mapped (generally speaking, many-valuedly) irreducibly and perfectly onto one another, it is necessary and sufficient that they have homeomorphic absolutes.

Moscow State University
named after M. V. Lomonosov

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## REFERENCES

[1] a) P. Alexandroff, Ann. of Math., **30**, 101 (1929); b) A. Kurosch, Comp. math., **2**, 471 (1935). [2] V. Ponomarev, DAN, **124**, No. 2, 268 (1959); Matem. sborn., **51**, 515 (1960). [3] V. Ponomarev, DAN, **132**, No. 6, 1269 (1960). [4] P. Alexandrov, V. Ponomarev, Sib. matem. zhurn., **1**, 1, 3 (1960). [5] P. Alexandrov, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **54** (1959).

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