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Abstract

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A GENERAL THEOREM ON THE FACTORIZATION OF MATRIX FUNCTIONS IN NORMED RINGS AND ITS APPLICATIONS

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1. Let F be an arbitrary closed set in the complex plane. By $K(F)$ we denote the ring of all rational functions whose poles do not belong to F.

We shall call a normed ring $C(F)$ of complex-valued continuous functions on F an R-ring if $K(F)$ is contained in $C(F)$ and is dense in it in the norm of the ring $C(F)$. It is easy to see that the bicompactum of maximal ideals of the R-ring $C(F)$ is homeomorphic to F. Hence, in particular, it follows that

$$\max_{\zeta\in F}|a(\zeta)| \leq \|a(\zeta)\|.$$

In what follows we shall assume that $C(F)$ is an R-ring. Let G_j $(j = 1, 2, \dots, \omega;\ \omega \leq \infty)$ be the set of all distinct connected components of the complement G of the set F. Denote by $\varkappa_j(r)$ $(j = 1, 2, \dots, \omega)$ the functional, defined on all functions $r(\zeta) \in K(F)$ for which $r(\zeta) \neq 0$ $(\zeta \in F)$, and equal to the difference between the number of poles and the number of zeros in G_j of the function $r(\zeta)$ (see (1)). The functionals $\varkappa_j(r)$ extend by continuity to the set $\mathcal{O}_C$ of all invertible elements of $C(F)$ and have the properties: 1) $\varkappa_j(r)$ is an integer for any $a \in \mathcal{O}_C$; 2) $\varkappa_j(a_1a_2) = \varkappa_j(a_1) + \varkappa_j(a_2)$ $(a_1, a_2 \in \mathcal{O}_C)$.

Denote by $C_{n\times n}(F)$ the ring of all matrices of functions of order n with elements from $C(F)$, and by $\mathcal{O}_{C;n}$ the set of all invertible matrix functions from the ring $C_{n\times n}(F)$, i.e. $\mathcal{O}_{C;n}$ consists of all $X(\zeta) \in C_{n\times n}(F)$ for which $\det X(\zeta) \neq 0$ $(\zeta \in F)$.

Theorem 1. *The general form of every functional $\varkappa(X)$, defined on the set of all $X \in \mathcal{O}_{C;n}$ and having properties 1) and 2), is given by the formula*

$$\varkappa(X) = \sum_j k_j\varkappa_j(X), \tag{1}$$

where k_j are arbitrary integers.

2. Suppose that the set G is not connected. Divide the set of all connected components of the domain G in some way into two classes: G_j^+ $(j = 1, 2, \dots, \omega^+;\ 0 < \omega^+ \leq \infty)$, G_j^- $(j = 1, 2, \dots, \omega^-;\ 0 < \omega^- \leq \infty)$. Put

$$G^{\pm} = \bigcup G_j^{\pm}; \qquad F^{\pm} = F \cup G^{\mp}; \qquad K^{\pm}(F) = K(F^{\pm}).$$

Denote by $C^+(F)$ ($C^-(F)$) the ring which is the closure of the ring $K^+(F)$ ($K^-(F)$) in the norm of the ring $C(F)$. Obviously, $C^{\pm}(F)$ is an R-ring on the set $F^{\pm}$.

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* The sum appearing on the right-hand side of equality (1) makes sense, since for each function $a \in \mathcal{O}_C$ only a finite number of the functionals $\varkappa_j$ can be nonzero.

A **left standard factorization** of a matrix-function $A(\zeta) \in \mathcal{O}_{C;n}$ (see (^{2})) is its representation in the form

$$A(\zeta) = A^+(\zeta)D(\zeta)A^-(\zeta) \quad (\zeta \in F), \tag{2}$$

where $A^{\pm}(\zeta) \in \mathcal{O}_{C^{\pm};n}$,

$$D(\zeta) = \left\| \left(\frac{\zeta - \lambda^+}{\zeta - \lambda^-} \right)^{\nu_j} \delta_{jk} \right\|_1^n, \tag{3}$$

$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ are certain integers, and $\lambda^{\pm}$ are certain points respectively from the domains $G^{\pm}$.

If in equality (2) the factors $A^{\pm}(\zeta)$ are interchanged, then the resulting factorization of the matrix-function $A(\zeta)$ is called a **right standard factorization**.

It is easy to see that if the matrix-function $A(\zeta)$ admits the factorization (2) with some pair of points $\lambda^{\pm} \in G^{\pm}$, then it admits the factorization (2) also with any other pair of points chosen arbitrarily in the corresponding domains $G^{\pm}$. In particular, if the domain F is bounded, $\zeta = 0 \in G^+$ and $\zeta = \infty \in G^-$, then equality (3) may be replaced by the equality

$$D(\zeta) = \left\| \zeta^{\nu_j} \delta_{jk} \right\|_1^n .$$

If the matrix-function $A(\zeta) \in \mathcal{O}_{C;n}$ admits a left (right) standard factorization, then the numbers $\nu_j (= \nu_j(A))$ are uniquely determined by the matrix-function A. These numbers are called, respectively according to the type of factorization, the **left** (**right**) **indices** of the matrix $A(\zeta)$.

Let us note that for any factorization

$$\sum_{j=1}^{n} \nu_j(A) = \sum_{j=1}^{\omega^+} \varkappa_j^+ (\det A(\zeta)),$$

where $\varkappa_j^+$ are all the functionals defined in Sec. 1 that correspond to the components G_j^+ $(j = 1, 2, \dots, \omega^+)$.

The following general theorem holds:

Theorem 2. In order that every matrix-function $A(\zeta) \in \mathcal{O}_{C;n}$ admit a left (right) standard factorization, it is necessary and sufficient that every function $x(\zeta) \in C(F)$ be representable in the form

$$x(\zeta) = x_+(\zeta) + x_-(\zeta) \quad (\zeta \in F), \tag{4}$$

where $x^\pm(\zeta) \in C^\pm(F)$.

This theorem is a generalization of the theorems from (2) on the factorization of matrix-functions for certain concrete R-rings.

In the proof of this theorem one device is used which, under other circumstances, was applied in the paper of G. F. Mandzhavidze and B. V. Khvedelidze (3).

All the principal results from (2) extend to the standard factorization of matrix-functions considered here in general R-rings. In particular, there are generalizations of the theorems on the general form of the factors $A^\pm(\zeta)$ in (2) (see (2), §7), the stability theorems for systems of indices ((2), §10), the theorems on the indices and factorization of triangular matrix-functions ((2), §11), and, for the case when F is the real line or the unit circle, the theorems on the factorization of Hermitian and pseudopositive matrix-functions ((2), §8).

3. We give one application of Theorem 2. Let V_t $(0 \le t < \infty;\ V_0 = I)$ be a strongly continuous semigroup of linear isometric operators acting in a separable Banach space $\mathfrak{B}$, and suppose $V_t\mathfrak{B} \neq \mathfrak{B}$ $(0 < t < \infty)$. Denote by T the generating operator of this semigroup.

As is known, T is a closed operator with a dense domain of definition, and all points λ with $\operatorname{Re}\lambda > 0$ are regular points of this operator.

For all points λ with $\operatorname{Re}\lambda < 0$ the relation $|(T - \lambda I)f| \ge |\operatorname{Re}\lambda|\,|f|$ $(f \in \mathfrak{B}_T)$ holds, and the quantity $\mathfrak{m} = \dim E/(T - \lambda I)\mathfrak{B}$ takes one and the same positive value.*

Suppose that each operator V_t has a bounded inverse on the left $V_t^{(-1)}$, which can be chosen so that the family $V_t = V_{-t}^{(-1)}$ $(-\infty < t \le 0)$ forms a strongly continuous semigroup and $|V_t| = 1$ $(-\infty < t \le 0)$. These conditions are, obviously, always satisfied when $\mathfrak{B} = \mathfrak{H}$ is a Hilbert space.

To each complex-valued finite continuous function $r(t)$ $(-\infty \le t \le \infty)$ we associate the operator A_r, defined by the equality

$$A_r f = \int_{-\infty}^{\infty} r(t) V_t f \, dt \qquad (f \in \mathfrak{B}).$$

Denote by $\mathfrak{K}$ the set of all operators of the form $\mu I + A_r$, where μ is an arbitrary complex number and r an arbitrary continuous finite function. Let the closure of the linear manifold $\mathfrak{K}$ in the operator norm be denoted by $\mathfrak{S}$. On the set $\mathfrak{K}$ we define a new multiplication by putting

$$(\mu_1 I + A_{r_1}) \circ (\mu_2 I + A_{r_2}) = \mu_1\mu_2 I + A_{\mu_1 r_2 + \mu_2 r_1 + r_1 * r_2},$$

where

$$(r_1 * r_2)(t) = \int_{-\infty}^{\infty} r_1(t-s) r_2(s)\, ds.$$

It is easily proved that for any pair of elements $\mu_1 I + A_{r_1}$, $\mu_2 I + A_{r_2} \in \mathfrak{K}$ the relation

$$\left\|(\mu_1 I + A_{r_1}) \circ (\mu_2 I + A_{r_2})\right\| \leq \left\|\mu_1 I + A_{r_1}\right\| \left\|\mu_2 I + A_{r_2}\right\|$$

holds.

It follows from this that the new multiplication operation can be extended by continuity to all pairs of elements of $\mathfrak{S}$. With this definition of multiplication, $\mathfrak{S}$ becomes a commutative normed ring. The bicompact of maximal ideals of this ring is homeomorphic to the compactified real axis. By $A(\zeta)$ $(-\infty \leq \zeta \leq \infty)$ we shall denote the function of the element A on the bicompact of maximal ideals of the ring $\mathfrak{S}$, and by $\mathfrak{C}$ the normed ring of all such functions with the ordinary operations and with the definition of the norm $\|A(\zeta)\| = |A|$. It is easy to see that $\mathfrak{C}$ is an R-ring on the real line.

By G^+ (G^-) we denote the upper (lower) open half-plane. The naturally defined subrings $\mathfrak{C}^+$ and $\mathfrak{C}^-$ are R-rings on $G^\pm$, respectively.

By $\mathfrak{B}_{n\times 1}$ we denote the direct sum of n copies of the space $\mathfrak{B}$. Then the ring $\mathfrak{S}_{n\times n}$ of matrices of order n with entries from $\mathfrak{S}$ consists of operators acting in $\mathfrak{B}_{n\times 1}$. To each such operator $A \in \mathfrak{S}_{n\times n}$ we naturally associate a matrix-function $A(\zeta)$ $(-\infty \leq \zeta \leq \infty)$ from $\mathfrak{C}_{n\times n}$.

Under the following additional assumptions: a) $\mathfrak{m} < \infty$; b) every function $x(\zeta) \in \mathfrak{C}$ representable in the form (4), the following holds:

Theorem 3. In order that the operator $A \in \mathfrak{C}_{n\times n}$ be normally solvable and that at least one of the equations $A\varphi = 0$ $(\varphi \in \mathfrak{B}_{n\times 1})$, $A^*\psi = 0$ $(\psi \in \mathfrak{B}^*_{n\times 1})$ have no more than a finite number of linearly independent solutions, it is necessary and sufficient that the condition

$$\det A(\zeta) \neq 0 \qquad (-\infty < \zeta < \infty). \tag{5}$$

be satisfied.

* The author owes this proposition to Yu. I. Lyubich.

If condition (5) is satisfied and the equality

$$\mathbf{A}^{-1}(\zeta) = A^{+}(\zeta)D(\zeta)A^{-}(\zeta) \qquad (-\infty < \zeta < \infty),$$

where

$$D(\zeta) = \left\| \left(\frac{\lambda - i}{\lambda + i} \right)^{\nu_i} \delta_{jk} \right\|_1^n,$$

gives the standard factorization of the matrix-function $\mathbf{A}^{-1}(\zeta)$, then the equation $\mathbf{A}\varphi = 0$ has exactly $\alpha = \sum_{\nu_j > 0} m\nu_j$ linearly independent solutions, and the equation $\mathbf{A}^{*}\psi = 0$ has exactly $\beta = -m\sum_{\nu_j < 0} \nu_j$ linearly independent solutions. In particular, the index of the operator $\mathbf{A}$ is determined by the equality

$$\varkappa(\mathbf{A}) = \alpha - \beta = -\frac{m}{2\pi} \left[\arg\det A(\zeta)\right]_{\zeta=-\infty}^{\zeta=\infty}.$$

If condition (5) is satisfied and, for the vector $y \in \mathfrak{B}_{n\times 1}$, the condition $\psi(y) = 0$ is satisfied, where ψ is an arbitrary solution of the equation $\mathbf{A}^{*}\psi = 0$, then one of the solutions of the equation $\mathbf{A}x = y$ is given by the equality

$$x = \mathbf{A}^{+}D\mathbf{A}^{-}y,$$

where $\mathbf{A}^{\pm}, D$ are the operators corresponding respectively to the matrix-functions $A^{\pm}(\zeta)\,(\in \mathcal{C}^{\pm}_{n\times n})$, $D(\zeta)\,(\in \mathcal{C}_{n\times n})$.

This theorem generalizes a number of theorems from $^{(2)}$ on systems of integral equations on a half-line with kernels depending on the difference of the arguments. From this theorem one can derive certain additions to the theorems indicated in $^{(2)}$.

Theorem 3 has a discrete analogue corresponding to the case in which the semigroup under consideration is discrete. The discrete analogue of Theorem 3 for $n = 1$ was formulated in $^{(1)}$*.

Because of lack of space, analogous generalizations are not given here for theorems on paired systems of integral equations and their transposes, for their discrete analogues, and for systems of one-dimensional singular integral equations.

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* As in (1), for the case $n = 1$, Theorem 3, in a slightly modified formulation, holds without assumptions a) and b).

Note: Figure translations are in progress. See original paper for figures.

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