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Abstract

Full Text

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ON SPACES WITH A POINT-COUNTABLE BASE*

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In the present note I prove (§ 1) that every bicompactum with a point-countable base is metrizable and that, on the other hand, there exist: 1) a nonmetrizable paracompactum (i.e., a paracompact Hausdorff space) with a point-countable base, 2) a Hausdorff (nonregular) hereditarily finally compact space with a point-countable base.

§ 1. **Theorem.** *Let R be a bicompact T_1-space and B a base such that for every point $x \in R$ the set B_x of all elements of the base B containing the point x has cardinality $\leq a$, where a is some cardinal number. Then* $\operatorname{card} B \leq a$.

Proof. We shall call a covering ω of the space R minimal if no proper part $\omega' \subset \omega$ covers R. It is easy to see that every covering ω of a bicompact space contains a minimal subcovering. Let V be the set of all possible minimal coverings of the bicompact space R composed of elements of the base B. Since every minimal covering is finite, the cardinality of the set V does not exceed the cardinality of B. But the cardinality of the set V cannot be less than the cardinality of B: this follows from the fact that for any $\sigma \in B$ there exists a covering $\omega \in V$ such that $\sigma \in \omega$. To prove the latter assertion, take some point $x \in \omega$. For each point $y \in R \setminus x$ choose an element $U(y)$ of the base B containing it and lying in $R \setminus x$. Denote by Ω the covering consisting of all the just selected $U(y)$ and of σ. Let ω be a minimal covering contained in Ω. Obviously, $\sigma \in \omega$. Thus, the cardinality of the set V is equal to the cardinality of B. Suppose that $\operatorname{card} B > a$. Then, if V_n denotes the set of all minimal coverings consisting of n distinct elements of the base, there exists an n_0 such that $\operatorname{card} V_{n_0} > a$. Let $k \leq n_0$, and let $\sigma_1, \dots, \sigma_k$ be arbitrary mutually distinct elements of the base B. Denote by $S_{\sigma_1 \sigma_2 \dots \sigma_k}$ the set of all those coverings $\omega \in V_{n_0}$ for which $\sigma_i \in \omega$ $(i = 1, 2, \dots, k)$.

Let x be an arbitrary point of the space R, and let B_x be the set of all elements of the base B containing it. Then the following equalities hold:

$$V_{n_0} = \bigcup_{\sigma \in B_x} S_\sigma. \tag{1}$$

If $\sigma_1, \dots, \sigma_k$ are any (pairwise distinct) elements of the base B not containing the point x, then

$$S_{\sigma_1\dots\sigma_k} = \bigcup_{\sigma \in B_x} S_{\sigma_1\dots\sigma_k\sigma}. \tag{2}$$

Let $x_1 \in R$ be an arbitrary point. There exists a $\sigma_1 \in B_{x_1}$ such that $\operatorname{card} S_{\sigma_1} > a$. Suppose that for $k < n_0$ such (pairwise distinct)

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* A system S of sets A_α is called point-countable if each $x \in \bigcup_\alpha A_\alpha$ belongs to no more than a countable number of sets $A_\alpha \in S$.

elements $\sigma_1, \dots, \sigma_k$ of the base B, such that $\operatorname{card} S_{\sigma_1\dots\sigma_k} > a$. Since $k < n_0$, there exists a point x_{k+1} not contained in any of the sets $\sigma_1, \dots, \sigma_k$. But then, by virtue of equality (2), there also exists a $\sigma_{k+1} \in Bx_{k+1}$ such that $\operatorname{card} S_{\sigma_1\dots\sigma_k\sigma_{k+1}} > a$. Moreover, obviously, σ_{k+1} does not coincide with any of $\sigma_1, \dots, \sigma_k$. As a result, for all $k < n_0$ the constructed sets $S_{\sigma_1\dots\sigma_k}$ have cardinality greater than a. In particular, the cardinality of the set $S_{\sigma_1\dots\sigma_{n_0}}$ exceeds the cardinal number a. But $S_{\sigma_1\dots\sigma_{n_0}} \subset V_{n_0}$, and therefore there exists only one cover $\omega \in S_{\sigma_1\dots\sigma_{n_0}}$, namely $\omega = \{\sigma_1, \sigma_2, \dots, \sigma_{n_0}\}$. The contradiction obtained proves the theorem.

§ 2. In this section a Hausdorff strongly paracompact nonmetrizable space with a point-countable base is constructed.

Let α be an ordinal number. As usual, denote by $W(\alpha)$ the set of all ordinal numbers $\beta < \alpha$. Let N be the set of all natural numbers. Denote by X_α the set of all mappings $x = x(\gamma)$, $\gamma < \alpha$, of the set $W(\alpha)$ into N (i.e., the set of all well-ordered sequences of type α whose elements are natural numbers: $\{x_1, x_2, \dots, x_\gamma, \dots\}$; $\gamma < \alpha$, $x_\gamma \in N$). Put

$$X = \bigcup_{1 < \alpha < \omega_0 + \omega_0} X_\alpha,$$

where $\omega = \omega_0$ is the first infinite ordinal number.

The length of an element $x \in X_\alpha$ will be called the ordinal number α. Thus every $x \in X$ has length $< \omega_0 + \omega_0$. Let $x \in X$, $y \in X$. We shall say that the element x is an extension of the element y if $\operatorname{len} x = \alpha > \beta = \operatorname{len} y$ and for every $\gamma < \beta$ we have $x(\gamma) = y(\gamma)$. Suppose the length of x is equal to α. Denote by $U_n(x)$ the set consisting of the point x and of all those $y \in X$ which are extensions of the element x and for which $y(\alpha) \geq n$. Then $B = \{U_n(x)\}_{n=1}^\infty$, $x \in X$, is a base of some topology on X. This follows from the fact that if $y \neq x$ and $y \in U_k(x)$, then for every n we have $U_n(y) \subset U_k(x)$.

Let us establish a number of properties of the base B.

1) If neither of the two elements x and y is an extension of the other, then $U_n(x) \cap U_m(y) = \Lambda$ for all n and m.

2) If x is an extension of y, but $x \notin U_m(y)$, then $U_n(x) \cap U_m(y) = \Lambda$ for all n.

Let us show that the base B is point-countable. If $x \in U_k(y)$, then x is an extension of y. The set of all such y that x extends y is at most countable. Hence the set of all such $U_k(y)$ for which $x \in U_k(y)$ is at most countable.

Let us show that into any cover ω one can inscribe a disjoint cover. For each point $x \in X_2$ there is a $U_n(x)$ lying in some element of the cover ω. The set of such neighborhoods, as x ranges over the set X_2, will be denoted by V_2. Then V_2 covers X_2 and is a disjoint system (for no two points $x, y \in X_2$ extend one another). Suppose that for every $\alpha < \beta < \omega_0 + \omega_0$ there has been constructed such a disjoint system of neighborhoods V_α of points of the set $\bigcup_{\gamma \leq \alpha} X_\gamma$, that:

1) V_α covers the set $\bigcup\limits_{\gamma \leq \alpha} X_\gamma$; 2) if $\alpha < \alpha' < \beta$, then $V_\alpha \subset V_{\alpha'}$.

Consider the set

$$G = \bigcup \{ U_k(x) : U_k(x) \in \bigcup_{\alpha < \beta} V_\alpha \}.$$

For each point $x \in X_\beta \setminus G$ there is a $U_k(x)$ lying in some element of the cover ω. Let $y \in X$ be an arbitrary point such that $U_k(y) \in \bigcup_{\alpha<\beta} V_\alpha$. Then $\operatorname{len} x > \operatorname{len} y$. Hence it follows that $U_k(x) \cap G = \Lambda$.

Since for any points $x, x' \in X_\beta \setminus G$ we have $\operatorname{len} x = \operatorname{len} x'$, it follows that $U_k(x) \cap U_s(x') = \Lambda$.

Denote then by V_β the sum of the set $\bigcup_{\alpha<\beta} V_\alpha$ and of all the neighborhoods $U_k(x)$ just considered, where x runs through the whole set $X_\beta \setminus G$. Then the system V_β is disjoint and covers $\bigcup_{\alpha \leqslant \beta} X_\alpha$, and moreover $V_\beta \supset V_\alpha$ for every $\alpha < \beta$. We have constructed V_β for every $\beta < \omega_0 + \omega_0$. The system $\bigcup_{\alpha < \omega_0 + \omega_0} V_\alpha$ is disjoint; it covers X and is inscribed in the cover ω.

This proves that X is a strongly paracompact (Hausdorff and, consequently, normal) space.

Let us show that X is not metrizable. Suppose the contrary. Then in the space X there exists a uniform base B_0 (in the sense of P. S. Aleksandrov ([1])). Let x_2 be an arbitrary point of X_2. There exists an $\sigma_2 \in B_0$ such that $x_2 \in \sigma_2$. Then there also exists $U_{n_2}(x_2) \subset \sigma_2$. Suppose that for all $k \leqslant m$ such $x_k, \sigma_k, U_{n_k}(x_k)$ have been constructed that $x_k \in X_k$, $\sigma_k \in B_0$,

$$U_{n_{k-1}}(x_{k-1}) \supset \sigma_k \supset U_{n_k}(x_k).$$

In particular, we have

$$U_{n_{m-1}}(x_{m-1}) \supset \sigma_m \supset U_{n_m}(x_m).$$

Choose some point $x_{m+1} \in X_{m+1}$, $x_{m+1} \in U_{n_m}(x_m)$. There exist such $\sigma_{m+1} \in B_0$ and $U_{n_{m+1}}(x_{m+1})$ that

$$U_{n_m}(x_m) \supset \sigma_{m+1} \supset U_{n_{m+1}}(x_{m+1}).$$

Thus $x_k, \sigma_k, U_{n_k}(x_k)$ are constructed by induction for every natural number k. Obviously, x_k is a continuation of x_{k-1}; but then there also exists a point $x \in X_{\omega_0}$ that is a continuation of x_k for every k. This point x, for every k, is contained in $U_{n_k}(x_k)$. Hence it follows that every neighborhood $U_n(x)$ of the point x lies in $U_{n_k}(x_k)$. Since $\sigma_k \supset U_{n_k}(x_k)$, we have $\sigma_k \supset U_n(x)$ for all numbers k, n. Thus, contrary to the assumption of uniformity of the base B_0, we have an infinite set of elements of this base $\{\sigma_k\}$, containing the point x and not forming a base at this point. The contradiction obtained proves the nonmetrizability of the space X.

§ 3. In this section we construct a hereditarily finally compact Hausdorff (but nonregular) space X with a point-countable base. The space has no countable base. Represent the interval $[0, 1]$ as the sum of a disjoint system, of cardinality $\aleph_1$, of nonempty everywhere dense sets X_α:

$$[0,1] = \bigcup_{\alpha<\omega_1} X_\alpha, \quad X_\alpha \cap X_\beta = \Lambda, \quad \text{where } \alpha \neq \beta.$$

Let p_1, p_2 be arbitrary rational numbers, $p_1 < p_2$. By (p_1, p_2) we denote the interval of the number line with endpoints p_1 and p_2. Let x be an arbitrary point of the interval $[0, 1]$, $x \in X_\alpha$. Define the neighborhood $U_{p_1p_2}(x)$ as the set

$$U_{p_1p_2}(x) = (p_1, p_2) \cap \bigcup_{\beta \geqslant \alpha} X_\beta.$$

1°. The neighborhood space R obtained in this way is Hausdorff. Indeed, for any two distinct points $x \in R$, $y \in R$, one can find disjoint rational intervals $(p_1', p_2'), (p_1'', p_2'')$, containing, respectively, the points x and y. Then, all the more,

$$U_{p_1'p_2'}(x) \cap U_{p_1''p_2''}(y) = \Lambda.$$

2°. The space R is hereditarily finally compact. Let
$\omega = \{\sigma_\gamma\}$ be a system of open sets. Then for every $x \in \tilde{\omega}$ (where $\tilde{\omega}$ denotes $\bigcup_\gamma \sigma_\gamma$) there is some $U_{p_1p_2}(x) \subset \sigma_\gamma \in \omega$.

The set of all pairs (p_1, p_2) is countable; therefore the set of those pairs for which there exists an $x \in R$ such that $U_{p_1p_2}(x) \subset \sigma_\gamma \in \omega$ is at most countable. Denote this set by $V = \{\gamma_i\}$, $i = 1, 2, ...,$ where $\gamma_i = (p_1^{(i)}, p_2^{(i)})$. For each i there exists an x_i such that $U_{\gamma_i}(x_i) \subset \sigma_\gamma \in \omega$. Let $x_i \in X_{\alpha_i}$. Take an ordinal $\alpha_0 < \omega_1$ exceeding all α_i.

Then

$$\bigcup_{i=1}^{\infty} U_{\gamma_i}(x_i) \supset \bigcup_{\beta \geq \alpha_0} X_\beta \cap \tilde{\omega}.$$

Since $X_\beta \cap \tilde{\omega}$ is finally compact in the usual topology of the interval $[0,1]$, for every $\beta < \alpha_0$ there exists a countable system of intervals $\{\sigma_i^\beta\}$ covering $X_\beta \cap \tilde{\omega}$ and inscribed in the cover ω_β generated by the system ω on $X_\beta \cap \tilde{\omega}$. Since the set of all $\beta < \alpha_0$ is at most countable, the set of all $\{\sigma_i^\beta\}$ is also at most countable.

Denote by $\sigma_i^{*\beta}$ the interval of the number line cut out from X_β by the set σ_i^β. Put $\sigma_i^{**\beta} = \sigma_i^{*\beta} \cap \bigcup_{\gamma \geq \beta} X_\gamma$. Then the countable system of open sets in R, $\{\sigma_i^{**\beta}\} \cup \{U_{\gamma_i}(x_i)\}$, covers all of $\tilde{\omega}$ and is inscribed in the system ω.

3°. The base $\{U_{p_1 p_2}(x)\}$ is point-countable. If $y \in U_{p_1 p_2}(x)$, and $y \in X_\alpha,\ X \in X_\beta$, then $\beta \leq \alpha$. But the set of all $\beta \leq \alpha$ is at most countable; consequently, the set of all $U_{p_1 p_2}(x) \ni y$ is at most countable.

4°. R has no countable base. Let B be any base of the space R. We shall show by contradiction that B is not a countable base, $B = \{\sigma_i\}$. Let $x_i \in \sigma_i,\ x_i \in X_{\alpha_i}$, and let α_0 exceed all α_i. Then, if $x_0 \in X_{\alpha_0}$, there exists no σ_i contained in $U_{p_1 p_2}(x_0)$, and the contradiction we need is obtained.

5°. The space R is nonregular. Let $X \subset R$ be the set of all rational numbers. There exists an α such that $X \subset \bigcup_{\beta < \alpha} X_\beta = A$. The set A is closed and cannot be separated by neighborhoods from any point $x \in R \setminus A$.

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REFERENCES

1. P. S. Aleksandrov, *Bull. Acad. Polon. Sci., Ser. Math.*, **8**, No. 3, 135 (1960).

Note: Figure translations are in progress. See original paper for figures.