MATHEMATICS

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Abstract

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EMBEDDING OF ZERO-DIMENSIONAL COMPACTA IN E^n

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1. A zero-dimensional compactum $A \subset E^n$, where E^n is n-dimensional Euclidean space, can be "wildly" embedded in E^n if $n \geqslant 3$, i.e., in such a way that there does **not** exist a homeomorphic mapping $f : E^n \to E^n$ for which $f(A) \subset E^1$, where E^1 is a line in E^n (1,2). A zero-dimensional compactum $A \subset E^n$ can be represented in the form

$$A = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} V_i^k; \qquad \overline{V}_i^k \cap \overline{V}_j^k = \Lambda, \quad i \neq j; \qquad \bigcup_{i=1}^{n_k} \overline{V}_i^k \subset \bigcup_{j=1}^{n_{k-1}} V_j^{k-1}, \tag{1}$$

where V_i^k are domains in E^n, whose diameters tend to zero as k increases, and $\overline{\mathcal{E}}$ is the closure of the set $\mathcal{E}$. It follows from (3) that if $n = 3$ and all $\overline{V}_i^k$ are topological balls, then A is tamely embedded in E^3, i.e., there exists a homeomorphism $f : E^3 \to E^3$ such that $f(A) \subset E^1$. We shall show that this holds for any n, and shall consider the question of obtaining an arbitrary zero-dimensional compactum in E^n by means of a homotopy $F_t : E^n \to E^n$, for every t, $a \leqslant t \leqslant b$, starting from a compactum lying on a line, and the identity mapping F_0.

Definition 1. A compactum $A \subset E^n$ is called **cellularly separated** in E^n if, in any of its neighborhoods V, one can inscribe a neighborhood U, the closure of which is the sum of a finite number of pairwise disjoint n-dimensional topological balls (n-elements):

$$A \subset U \subset \overline{U} \subset V; \qquad U = \bigcup_{r=1}^{m} U_r; \quad \overline{U}_r \cap \overline{U}_{r'} = \Lambda, \quad r \neq r'; \quad \overline{U}_r \text{ is an } n\text{-element.}$$

Definition 2. A **pseudo-isotopy** Φ_t, $0 \leqslant t \leqslant 1$, of a space X onto itself is a homotopy taking the identity mapping Φ_0 into a continuous mapping $\Phi_1 : X \to X$ such that, for every $t < 1$, Φ_t is a homeomorphic mapping of X onto itself.

Theorem 1. *For every cellularly separated zero-dimensional compactum A in E^n there exists an isotopy $F_t : E^n \to E^n$, $0 \leqslant t \leqslant 1$, from the identity mapping F_0 such that $F_1(A)$ lies on a line.*

Theorem 2. *For an arbitrary zero-dimensional compactum A in E^n there exists a compactum C, lying on a line, and a pseudo-isotopy Φ_t, $0 \leqslant t \leqslant 1$, of the space E^n onto itself such that $\Phi_1(C) = A$, and the mapping Φ_1 is a homeomorphism on the set $E^n \setminus \Phi_1^{-1}(A)$ and on the compactum C.*

In the case when A has no isolated points, the Cantor perfect set may be taken as C.

2. Lemmas. For all the isotopies considered below, F_0 is the identity mapping; therefore we shall not stipulate this.

Lemma 1. *Let δ^n, δ'^n be two closed balls in E^n with centers O, O' and radii r, r', and let U be a domain containing $\delta^n \cup \delta'^n$. There exist-*

there is an isotopy $F_t : E^n \to E^n$ from the identity F_0 such that $F_1(\delta^n) = \delta'^n$ and $F_t(x) = x$, if $x \in E^n \setminus U$, $0 \leq t \leq 1$.

If $O = O'$, then there exists an isotopy F_t carrying linearly each radius of the ball δ^n into a radius of the ball δ'^n and fixed outside an arbitrary ball concentric with δ^n and δ'^n and containing $\delta^n \cup \delta'^n$.

If the points O and O' are distinct, then they can be joined in U by a polygonal line

$$l = \bigcup_{i=1}^{k} l_i,$$

where l_i are the links of l. For each l_i choose a cylindrical neighborhood C_i so that

$$l_i \subset C_i \subset \overline{C}_i \subset U; \qquad \overline{C}_i \cap \overline{C}_j = \Lambda, \quad \text{if } |i-j| > 1.$$

Each vertex $a_i = l_i \cap l_{i+1}$ is contained in $C_i \cap C_{i+1}$. Choose a number ρ such that for every i the ball δ_i of radius ρ with center at the point a_i is contained in $C_i \cap C_{i+1}$. First construct an isotopy F_t, $0 \leq t \leq 1/(k+2)$, so that $F_0(x) \equiv x$, $F_{1/(k+2)}(\delta'^n) = \delta_0^n$, where δ_0^n is the ball of radius ρ with center O, and $F_t(x) = x$ outside a small neighborhood $\delta'^n \cup \delta_0^n$.

For each i an isotopy φ_t^i is constructed for $i/(k+2) \leq t \leq (i+1)/(k+2)$, piecewise linearly carrying into itself every straight line parallel to l_i, such that $\varphi_t^i(x) = x$, if $x \in E^n \setminus C_i$; $\varphi^i_{(i+1)/(k+2)}(\delta^n_{i-1}) = \delta^n_i$; $\varphi^i_{i/(k+2)}(x) \equiv x$. Finally, for $(k+1)/(k+2) \leq t \leq 1$ the isotopy φ_t^{k+1} is fixed outside a small neighborhood $\delta'^n \cup \delta_k^n$, and $\varphi_1^{k+1}(\delta_k^n) = \delta'^n$.

The isotopy F_t is defined successively by the equalities

$$F_t(x) = \varphi_t^i F_{i/(k+2)}(x), \qquad i/(k+2) \le t \le (i+1)/(k+2), \quad i = 1, 2, \dots, k+1.$$

Since all φ_t^i are fixed outside U, we also have $F_t(x) = x$, if $x \in E^n \setminus U$, $0 \le t \le 1$. By construction $F_1(\delta^n) = \delta'^n$.

Denote by $\operatorname{int}\mathcal{E}$ the set of interior points of $\mathcal{E}$, and $\partial\mathcal{E} = \overline{\mathcal{E}} \setminus \operatorname{int}\mathcal{E}$.

Lemma 2. *For any n-element $Q^n \subset E^n$, compactum $K \subset \operatorname{int} Q^n$, ball δ^n, and domain $U \supset Q^n \cup \delta^n$, there exist an n-element $Q'^n \subset \operatorname{int} Q^n$, containing K, and an isotopy F_t from the identity F_0, fixed on $E^n \setminus U$, such that $F_1(Q'^n) = \delta^n$.*

Let φ be a homeomorphic mapping of the ball

$$B^n \left(\sum_{i=1}^n x_i^2 \le 1 \right)$$

onto Q^n, and let $O' = \varphi(O)$, where O is the center of B^n. By Lemma 1 one may assume that $\delta^n \subset \operatorname{int} Q^n$ and O' is the center of δ^n. Then $\varphi^{-1}(\delta^n)$ is an n-element in B^n, and $O \subset \operatorname{int}\varphi^{-1}(\delta^n)$. In B^n choose two n-dimensional balls with center O:

$$B' \subset \operatorname{int}\varphi^{-1}(\delta^n); \qquad B'' \supset \varphi^{-1}(\delta^n) \cup \varphi^{-1}(K). \tag{2}$$

There exists an isotopy $h_t : B^n \to B^n$, fixed on ∂B^n, such that $h_1(B'') = B'$.

$\theta = h_1^{-1}\varphi^{-1}(\delta^n)$ is an n-element and, by (2), $\varphi^{-1}(K) \subset \theta$ and $B' \subset \theta$. $F_t = \varphi h_t \varphi^{-1}$ is an isotopy of Q^n onto itself, fixed on ∂Q^n. Put $Q'^n = \varphi(\theta)$; then $K \subset Q'^n$. It is easy to verify that $F_1(Q'^n) = \delta^n$. Setting $F_t(x) = x$ for $x \in E^n \setminus Q^n$, we satisfy the conditions of the lemma.

3. Proof of Theorem 1. All $\overline{V}_i^k$ in this case are n-elements. Let Δ be a ball,

$$\bigcup_{i=1}^{n_1} \overline{V}_i^1 \subset \Delta.$$

Choose n_1 points on the segment $E^1 \cap \Delta$ and balls $\delta_i^1 \subset \Delta$ with centers at them, such that

$$\delta_i^1 \cap \delta_j^1 = \Lambda, \quad i \ne j; \qquad \left(\bigcup_{i=1}^{n_1} \overline{V}_i^1 \right) \cap \left(\bigcup_{i=1}^{n_1} \delta_i^1 \right) = \Lambda. \tag{3}$$

Join in Δ each $\overline{V}_i^1$ by a polygonal line l_i with δ_i^1 so that $d_i \cap d_j = \Lambda$, where $d_i = \overline{V}_i^1 \cup l_i \cup \delta_i^1$, and for each d_i choose a neighborhood $U_i \subset \Delta$ such that

$\overline{U}_i \cap \overline{U}_j = \Lambda$, $i \neq j$. According to Lemma 2, in each $\overline{V}_i^1$ one can choose an n-element W_i^1 so that

$$\left(\bigcup_j \overline{V}_j^2\right) \cap V_i^1 \subset \operatorname{int} W_i \subset V_i^1, \tag{4}$$

and construct an isotopy $F_t^i : \overline{U}_i \to \overline{U}_i$, $0 \leqslant t \leqslant 1/2$, fixed on ∂U_i, for which $F_{1/2}^i(W_i) = \delta_i^1$. Put

$$F_t(x) = F_t^i(x), \quad x \in \overline{U}_i, \qquad F_t(x) = x, \quad x \in E^n \setminus \bigcup_{i=1}^n U_i; \quad 0 \leqslant t \leqslant 1/2. \tag{5}$$

Suppose that the isotopy F_t has been constructed for $t \leqslant (k-1)/k$; δ_i^r, $r \leqslant k-1$, are balls with centers on E^1, and

$$F_t(\overline{V}_i^r) \subset \delta_j^{r-1}, \quad \text{if } \overline{V}_i^r \subset V_j^{r-1}, \quad \frac{r-1}{r} \leqslant t \leqslant \frac{k-1}{k};$$

$$F_t(x) = F_{(r-1)/r}(x), \quad \text{if } t > \frac{r-1}{r} \quad \text{and} \quad x \in E^n \setminus \bigcup_{j=1}^{n_{r-1}} V_j^{r-1} \tag{6}$$

(this is true for $k = 2$); construct F_t for $t \leqslant k/(k+1)$. Choose in each $\delta_j^{k-1} \setminus F_{(k-1)/k}\left(\bigcup \overline{V}_i^k\right)$ disjoint balls δ_i^k with centers on E^1 for all $\overline{V}_i^k \subset V_j^{k-1}$. Having chosen $W_i^k \subset F_{(k-1)/k}(V_i^k)$, one on each δ_j^{k-1}, similarly to F_t for $0 \leqslant t \leqslant 1/2$ in Δ, constructs an isotopy $\varphi_t^j : \delta_j^{k-1} \to \delta_j^{k-1}$, $(k-1)/k \leqslant t \leqslant k/(k+1)$, fixed on $\partial\delta_j^{k-1}$, such that $\varphi_{k/(k+1)}^j(W_i^k) = \delta_i^k$; $\varphi_{k/(k+1)}^j F_{(k-1)/k}(\overline{V}_s^{k+1}) \subset \operatorname{int} \delta_i^k$, if $\overline{V}_s^{k+1} \subset V_i^k$; and put

$$F_t(x) = \varphi_t^j F_{(k-1)/k}(x), \quad \text{if } x \in F_{(k-1)/k}^{-1}(\delta_j^{k-1});$$

$$F_t(x) = F_{(k-1)/k}(x), \quad \text{if } x \in E^n \setminus \bigcup_j F_{(k-1)/k}^{-1}(\delta_j^{k-1});$$

$$\frac{k-1}{k} \leqslant t \leqslant \frac{k}{k+1}.$$

Thus F_t is defined for $t < 1$, and (6) is satisfied for every k. Put $F_1(x) = F_{(k-1)/k}(x)$, if $x \in V_{i_k}^k \setminus \bigcup_j \overline{V}_j^{k+1}$, $F_1\left(\bigcap_{k=1}^\infty V_{i_k}^k\right) = \bigcap_{k=1}^\infty \delta_{i_k}^k$, $k = 1, 2, \ldots$.

By virtue of (6), F_1 is a homeomorphism for each t, $F_1(A) = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} \delta_i^k \subset E^1$, and F_t converges uniformly to F_1 as $t \to 1$; hence $F_t,\ 0 \leqslant t \leqslant 1$, is an isotopy.

4. Proof of Theorem 2. The closed regions $\overline{V}_i^k$ in formula (1) may fail to be n-elements. On the line E^1 place a compactum

$$C = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} \delta_i^k; \quad \delta_i^k \cap \delta_{i'}^k = \Lambda, \quad i \neq i'; \quad \delta_i^k \subset \operatorname{int} \delta_j^{k-1},$$

if $\overline{V}_i^k \subset V_j^{k-1}$, where all δ_i^k are balls with centers on E^1, $\operatorname{diam} \delta_i^k \to 0$ as $k \to \infty$, and condition (3) is satisfied. Choose in each $V_i^1 \setminus \bigcup \overline{V}_j^2$ a ball π_i^1. As above, construct n_i^1 regions U_i^1 so that $\pi_i^1 \cup \delta_i^1 \subset U_i^1$, $\overline{U}_i^1 \cap \overline{U}_j^1 = \Lambda$, $i \neq j$, and an isotopy $\Phi_t,\ 0 \leqslant t \leqslant 1/2$, fixed on $E^n \setminus \bigcup_{i=1}^{n_1} U_i^1$, satisfying the condition

$$\Phi_t(\overline{U}_i^1) = \overline{U}_i^1; \quad \Phi_{1/2}(\delta_i^1) = \pi_i^1.$$

Let π_j^{r-1} be a ball in $V_j^{r-1} \setminus \bigcup \overline{V}_i^r$; Φ_t has been constructed

for $t \leq (k-1)/k$ and for $r \leq k$:

$$\begin{aligned} &\text{a)} \quad \Phi_{(r-1)/r}(\delta_i^r) \subset \pi_j^{r-1} \subset V_j^{r-1} \setminus \bigcup \overline{V}_s^r, \quad \text{if } \delta_i^r \subset \delta_j^{r-1}; \\ &\text{b)} \quad \Phi_t(x) = \Phi_{(r-1)/r}(x), \quad \text{if } t > \frac{r-1}{r} \text{ and } \Phi_{(r-1)/r}(x) \in E^n \setminus \bigcup_{j=1}^{n_{r-1}} V_j^{r-1}. \end{aligned} \tag{7}$$

\end{equation}

To construct Φ_t for $t \leq k/(k+1)$, choose in each $V_i^k \setminus \bigcup \overline{V}_m^{k+1}$ a ball π_i^k; then, for all V_i^k, construct domains U_i^k so that

$$\begin{gathered} \Phi_{(k-1)/k}(\delta_i^k) \cup \pi_i^k \subset U_i^k; \qquad U_i^k \subset V_j^{k-1}, \quad \text{if } V_i^k \subset V_j^{k-1}, \\ \overline{U}_i^k \cap \overline{U}_{i'}^k = \Lambda, \quad i \neq i'. \end{gathered} \tag{8}$$

By Lemma 2 one can choose in each n-element $\Phi_{(k-1)/k}(\delta_i^k)$ an n-element W_i^k for which

$$\Phi_{(k-1)/k}\left[\delta_i^k \cap \left(\bigcup_{m=1}^{n_{k+1}} \delta_m^{k+1}\right)\right] \subset W_i^k \subset \Phi_{(k-1)/k}(\delta_i^k), \tag{9}$$

and construct an isotopy $\varphi_t,\ (k-1)/k \leq t \leq k/(k+1)$, such that

$$\begin{gathered}\varphi_t(\overline{U}_i^k)=\overline{U}_i^k,\qquad \varphi_t(x)=x,\quad \text{if } x\in E^n\setminus\bigcup_{i=1}^{n_k}U_i^k,\\ \varphi_{k/(k+1)}(W_i^k)=\pi_i^k.\end{gathered}\tag{10}$$

Put

$$\Phi_t(x)=\varphi_t\Phi_{(k-1)/k}(x),\qquad \frac{k-1}{k}\le t\le\frac{k}{k+1}.\tag{11}$$

Then (7) is satisfied for $t\le k/(k+1)$. Note that π_i^k may fail to be contained in π_j^{k-1} when $V_i^k\subset V_j^{k-1}$.

Thus Φ_t is constructed for $t<1$. From (8) and (10) it follows that if $\Phi_{(k-1)/k}(x)\in V_j^{k-1}$, then also, for $t>(k-1)/k$, $\Phi_t(x)\in V_j^{k-1}$; by virtue of this and (7b), the set of homeomorphisms Φ_t, $t<1$, converges uniformly as $t\to1$ to a continuous mapping $\Phi_1:E^n\to E^n$, $\Phi_1(x)=\lim_{t\to1}\Phi_t(x)$. In this case

$$\begin{gathered}\Phi_1(x)=\Phi_t(x)\big|_{t\ge(k-1)/k}=\Phi_{(k-1)/k}(x),\quad \text{if } x\in\Phi_{(k-1)/k}^{-1}\left[E^n\setminus\bigcup_{i=1}^{n_{k-1}}\overline{V}_i^{k-1}\right];\\ \Phi_1(x)\in V_j^{k-1},\quad \text{if } x\in\Phi_{(k-1)/k}^{-1}(V_j^{k-1});\\ \Phi_1\left(\bigcap_{k=1}^{\infty}\delta_{i_k}^k\right)=\lim_{k\to\infty}\pi_{i_k}^k=\bigcap_{k=1}^{\infty}V_{i_k}^k,\qquad \delta_{i_k}^k\subset\delta_{i_{k-1}}^{k-1}.\end{gathered}\tag{12}$$

It follows that Φ_1 is a homeomorphism on the compactum C and $\Phi_1(C)=A$. But there may be points $x\in E^n\setminus C$ such that, for every k, $\Phi_{(k-1)/k}(x)\in V_{i_k}^k$. By virtue of (8) and (10), $V_{i_k}^k\subset V_{i_{k-1}}^{k-1}$, and therefore $\Phi_1(x)\in C$. By virtue of (12), Φ_1 is a homeomorphism on $E^n\setminus\Phi_1^{-1}(A)$.

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