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Abstract

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MATHEMATICS

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THE CONNECTION BETWEEN THE INVERSE PROBLEM AND COMPLETENESS OF EIGENFUNCTIONS

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1°. In the present article, in contrast to ([1]), a different approach is proposed to the inverse problem of spectral analysis for ordinary differential operators, based on an observation of Borg ([2]) (see also ([3], [4])). This approach can serve only to clarify the question of the unique determination of the solution of the inverse problem by reducing this question to a certain question of completeness. In its main features, the proposed approach is a procedure consisting of the following stages:

1) Generalizing the well-known observation of Borg ([2]), we reduce the question of unique determination to the question of completeness of certain products of eigenfunctions (see formula (8)).

2) We construct an auxiliary ordinary linear differential operator with spectral parameter λ, whose eigenfunctions are these products (see formulas (4), (9), and (11)).

3) Ultimately, generally speaking, the question of uniqueness of the solution of the inverse problem will be reduced to the investigation of the completeness of the eigenfunctions of the constructed auxiliary operator (see Lemma 1).

Although all stages 1)–3) of the procedure are of a general character, its concrete implementation has been carried out by us only in the case of equations for vector functions.

2°. Let us formulate the inverse problem and the uniqueness theorem in the case of equations for vector functions. For $k \geqslant 1$, let E^k denote the k-dimensional coordinate complex Euclidean space, and $(u, v)_k$ the scalar product of vectors $u, v \in E^k$.

Consider the equation

$$\frac{dx}{dt} = [\lambda a(t) + A(t)]x, \tag{1}$$

where λ is a complex number; the vector function $x(t)$ $(0 \leqslant t \leqslant 1)$ takes values in E^n; the matrices $a(t)$ and $A(t)$ are summable in t, with $a(t)$ diagonal and $A(t)$ containing no diagonal elements, i.e.

$$a(t) = \|a_i(t)\delta_{ij}\|_{1\le i,j\le n}, \qquad A(t) = \|A_{ij}(t)\|_{1\le i,j\le n}, \qquad A_{ii}(t) \equiv 0 \quad (1 \le i \le n). \tag{2}$$

For given λ, $a(t)$, and $A(t) = A$, we define the matrix $W_A(\lambda)$ by the equality

$$W_A(\lambda)x(0) = x(1) \tag{3}$$

for all $x(t)$ satisfying (1).

Fix $a(t)$. We consider the inverse problem of determining $A(t)$ from the given $W_A(\lambda)$. Problems close to this were considered by V. P. Potapov ([5]), L. A. Sakhnovich ([6]), and M. S. Brodskii ([7]).

Theorem 1. Let, in equation (1), $a(t)$ be fixed and continuous, and suppose there exists a complex $\lambda_0 \neq 0$ such that

$$\operatorname{Re}\lambda_0 a_1(t) < \cdots < \operatorname{Re}\lambda_0 a_n(t) \quad (0 \le t \le 1).$$

Then the solution of the inverse problem for equation (1) is uniquely determined; moreover, if $W_A(\lambda)$ and $W_B(\lambda)$ are defined by relations of the form (3), and the matrix $W_A^{-1}(\lambda)W_B(\lambda)$ is diagonal for all λ, then

$$A(t) = B(t) \quad (0 \le t \le 1).$$

Theorem 1 remains valid also when the fixed $a(t)$ is not continuous, but there exist real constants C' and C'' such that

$$C' < \arg[a_{i+1}(t) - a_i(t)] < C'' < C' + \pi \quad (1 \le i \le n-1)$$

for almost all t.

The proof of Theorem 1 is carried out, in accordance with the procedure described in 1°, as follows. The eigenfunctions of the auxiliary operator satisfy an equation of the form

$$\frac{dz_1}{dt} = [\lambda c(t) + M_1(t)]z_1 + M_2(t)z_2,$$

$$\frac{dz_2}{dt} = M_3(t)z_1, \qquad z_2(0) = 0, \tag{4}$$

where $z_1(t)$ and $z_2(t)$ $(0 \le t \le 1)$ take their values respectively in E^{N_1} and E^{N_2}, and the matrices $c(t)$ and $M_\nu(t)$ $(\nu = 1, 2, 3)$ have the corresponding numbers of rows and columns and are summable in t; moreover, $c(t)$ is diagonal, i.e.

$$c(t) = \|c_\alpha(t)\delta_{\alpha\beta}\|_{1\le\alpha,\beta\le N_1}.$$

We shall say that equation (4) has a complete system of solutions if there does not exist a nonzero summable vector-function $k(t)$ with values in E^{N_1} such that

$$\int_0^1 (z_1(t), k(t))_{N_1}\, dt = 0 \tag{5}$$

for all λ and all $z_1(t)$ satisfying (4).

The application of the procedure described in 1° to the proof of Theorem 1 is indicated by the lemma:

Lemma 1. Let $a_1(t), \dots, a_n(t)$ be fixed and such that, for arbitrary $M_\nu(t)$ $(\nu = 1, 2, 3)$, equation (4) has a complete system of solutions in the case where $N_1 = n(n-1)$, $N_2 = n$, and the sequence $c_1(t), \dots, c_{N_1}(t)$ is the collection, written in a definite order, of functions of the form

$$a_i(t) - a_j(t) \quad (1 \le i, j \le n;\ i \ne j).$$

Then the solution of the inverse problem for equation (1) is uniquely determined; moreover, if for all λ the matrix $W_A^{-1}(\lambda)W_B(\lambda)$ is diagonal, then

$$A(t) = B(t) \quad (0 \le t \le 1).$$

Proof. Introduce the matrix

$$\Delta(t) = B(t) - A(t) = \|\Delta_{ij}(t)\|_{1\le i,j\le n}.$$

Since $\Delta_{ii}(t) \equiv 0$ $(1 \le i \le n)$, it is enough to prove that $\Delta_{ij}(t) \equiv 0$ for $i \ne j$. Let the vector-functions

$$y(t) = \{y_i(t)\}_{1\le i\le n} \quad \text{and} \quad p(t) = \{p_i(t)\}_{1\le i\le n}$$

satisfy the equations

$$\frac{dy}{dt} = [\lambda a(t) + B(t)]y,$$

$$\frac{dp}{dt} = -[\bar\lambda\, \overline{a}(t) + \overline{A}(t)]p, \qquad y_i(0)p_i(0) = 0 \quad (1 \le i \le n). \tag{6}$$

Denote by

$$Z(t) = y(t) \otimes \overline{p(t)} = \|Z_{ij}(t)\|_{1\le i,j\le n} = \|y_i(t)\overline{p_j(t)}\|_{1\le i,j\le n}. \tag{7}$$

tensor product of the vectors $y(t)$ and $\overline{p(t)} = \{\overline{p_i(t)}\}_{1\le i\le n}$. From the conditions of the lemma (1), (3), (6), and (7) we obtain that

$$\int_0^1 \sum_{i \neq j} \Delta_{ji}(t) Z_{ij}(t)\, dt = 0 \tag{8}$$

for all $Z_{ij}(t)$, λ, $y(t)$, and $p(t)$ satisfying (6) and (7). Differentiating (7) and taking (6) into account, we find that

$$\frac{dZ}{dt} = \lambda[a(t)Z - Za(t)] + B(t)Z - ZA(t), \qquad Z_{ii}(0) = 0 \quad (1 \leq i \leq n). \tag{9}$$

Introduce the vector-function $z_1(t)$ with values in E^{N_1} ($N_1 = n(n-1)$) and components $Z_{ij}(t)$ ($1 \leq i, j \leq n;\ i \neq j$), arranged in a definite order, and the vector-function $z_2(t)$ with values in E^n and components $Z_{ii}(t)$ ($1 \leq i \leq n$). It follows from (9) that $z_1(t)$ and $z_2(t)$ satisfy an equation of the form (4) with certain values $M_\nu(t)$ ($\nu = 1, 2, 3$). For this equation the conditions stated in the formulation of the lemma are fulfilled, and therefore it has a complete system of solutions. Now write relation (8) in the form (5), where the vector-function $k(t)$ has components $\Delta_{ji}(t)$ ($1 \leq i, j \leq n,\ i \neq j$), arranged in a definite order. Since the equation of the form (4) for the vector-functions $z_1(t)$ and $z_2(t)$ has a complete system of solutions, it follows that $k(t) \equiv 0$ and $\Delta_{ij}(t) = 0$ for $i \neq j$. Lemma 1 is proved.

3°. The proof of Theorem 1 is completed with the aid of the completeness theorem for equation (4), formulated as follows.

Let S denote the unit circle $|\lambda| = 1$ in the complex λ-plane. Further, let $d = d(t)$ be an arbitrary complex-valued function on $0 \leq t \leq 1$. Denote by $P(d)$ the set of points $\lambda_0 \in S$ such that: a) for some t_{λ_0} ($0 \leq t_{\lambda_0} \leq 1$), $\operatorname{Re} \lambda_0 d(t) \leq 0$ for almost all $t \leq t_{\lambda_0}$ and $\operatorname{Re} \lambda_0 d(t) \geq 0$ for almost all $t \geq t_{\lambda_0}$; b) there is no interval on which $\operatorname{Re} \lambda_0 d(t) = 0$ almost everywhere.

Theorem 2. *Equation (4) has a complete system of solutions if the functions* $c_\alpha = c_\alpha(t)$ ($1 \leq \alpha \leq N_1$) *are such that the set*

$$P = \bigcap_{1 \leq \alpha \leq N_1} P(c_\alpha)$$

is not contained in any semicircle. The conditions of Theorem 2 are fulfilled if there exist real constants C' *and* C'' *such that* $C' < C'' < C' + \pi$ *and, for each* α ($1 \leq \alpha \leq N_1$), *either* $C' < \arg c_\alpha(t) < C''$ *almost everywhere on* $0 \leq t \leq 1$, *or* $C' + \pi < \arg c_\alpha(t) < C'' + \pi$ *almost everywhere.*

Theorem 2 is proved using a theorem of Phragmén–Lindelöf type.

4°. In view of the application to other equations of the general procedure described in 1°, we shall establish an assertion generalizing relation (9). Consider a system of linear ordinary differential equations containing the parameter λ:

$$x_i^{(q_i)}(t,\lambda) = \sum_{j=1}^{L} \sum_{q=0}^{q_j-1} a_{qji}(t,\lambda)\, x_j^{(q)}(t,\lambda) \qquad (1 \le i \le L) \tag{10}$$

($F^{(q)}(t,\lambda)$ denotes the q-th derivative with respect to t.)

By a polynomial we shall mean a function

$$f(t,\lambda) = P(x_1(t,\lambda), \dots, x_1^{(q_1-1)}(t,\lambda); \dots; x_L(t,\lambda), \dots, x_L^{(q_L-1)}(t,\lambda)),$$

where $P(u_{11}, \dots, u_{1q_1}; \dots; u_{L1}, \dots, u_{Lq_L})$ is a polynomial in all its variables.

The following assertion holds:
Let $x_1(t,\lambda), \dots, x_L(t,\lambda)$ satisfy equation (10). For any polynomials $f_1(t,\lambda), \dots, f_r(t,\lambda)$ one can choose a number $R > r$ and polynomials $f_{r+1}(t,\lambda), \dots, f_R(t,\lambda)$ such that $f_1, \dots, f_R$ satisfy some linear system of ordinary differential equations containing the parameter λ:

$$f_k' = \sum_{l=1}^{R} b_{kl}(t,\lambda) f_l \quad (1 \le k \le R). \tag{11}$$

If the parameter λ enters system (10) linearly, then λ also enters system (11) linearly.

For example, if $\varphi(t,\lambda)$ $(0 \le t \le 1)$ is a solution of the Sturm–Liouville equation

$$\varphi'' = [\lambda - q(t)]\varphi$$

and

$$\varphi(0) = \varphi(1) = 0,$$

then

$$f_1(t,\lambda) = \varphi^2(t,\lambda), \quad f_2(t,\lambda) = \varphi(t,\lambda)\varphi'(t,\lambda), \quad f_3(t,\lambda) = \varphi'^2(t,\lambda)$$

satisfy the system of equations

$$f_1' = 2f_2, \qquad f_2' = [\lambda - q(t)]f_1 + f_3, \qquad f_3' = 2[\lambda - q(t)]f_2,$$

with

$$f_1(0,\lambda) = f_2(0,\lambda) = f_1(1,\lambda) = f_2(1,\lambda) = 0.$$

Hence it follows easily that

$$f_1''' = 4[\lambda - q(t)]f_1' - 2q'(t)f_1,$$

$$f_1(0,\lambda) = f_1'(0,\lambda) = f_1(1,\lambda) = f_1'(1,\lambda) = 0.$$

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Note: Figure translations are in progress. See original paper for figures.