MATHEMATICS

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Abstract

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ON THE PROBLEM IN THE WHOLE SPACE FOR A CERTAIN CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

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1. It is known that any partial differential equation with constant coefficients

$$P\left(\frac{1}{i}\frac{\partial}{\partial x_1},\dots,\frac{1}{i}\frac{\partial}{\partial x_n}\right)u(x_1,\dots,x_n)=f(x_1,\dots,x_n) \tag{1}$$

has infinitely many different solutions. However, in some cases one can impose such conditions on the behavior of the function $u(x)$ as $x \to \infty$ which, for every right-hand side $f(x)$ belonging to a certain class, determine a unique solution of equation (1). In the case when the characteristic polynomial $P(\sigma_1,\dots,\sigma_n)$ of equation (1) has no real zeros, the question of such conditions was solved in the works of V. P. Palamodov ([4],[5]). The presence of real zeros of the polynomial $P(\sigma_1,\dots,\sigma_n)$ substantially changes the picture. We shall first dwell on questions of uniqueness.

2. Let B be a certain normed space consisting of functions (ordinary or generalized) such that the space D of all finite infinitely differentiable functions is contained in B and is dense there. We shall assume that convergence in the space D entails convergence in the norm of the space B. In this case every linear continuous functional on B is a generalized function, and one can speak of the support of this functional.

Definition. A closed set $E \subset R^n$ will be called **thin** relative to B if there exists no nontrivial element of B' with support in E.

If, for example, we take as B the space L_q, $1 \le q < \infty$, then E will be thin if and only if the measure of E is zero. It is easy to see that E is thin relative to B if and only if the functions from D with supports lying outside E are dense in the space B.

If multiplication by any function $\varphi \in D$ is a continuous operation in B, then one can assert that a closed bounded set E is thin relative to B if and only if there exists a sequence $\varphi_\nu \in D$ such that each φ_ν is equal to one in some neighborhood of the set E and $\|\varphi_\nu\|_B \to 0$. An unbounded set is thin if and only if it intersects each ball in a thin set.

Using these considerations, it is easy to show that a linear subspace R^j of dimension j of the space R^n is thin relative to the spaces $H_p^r(R^n)$ of S. M. Nikol' skii and $W_p^r(R^n)$ of S. L. Sobolev if

$$r - \frac{n}{p} + \frac{j}{p} < 0$$

(here $r > 0$ is arbitrary). If

$$r - \frac{n}{p} + \frac{j}{p} > 0,$$

then, as follows from the known embedding theorems, R^j is not thin (6).

For the spaces $W_2^r(R^n)$, Hörmander and Lions established that R^j is thin with respect to $W_2^r(R^n)$ if and only if $r - \frac{n}{p} + \frac{j}{p} \leq 0$ (2).

Since under a sufficiently smooth change of coordinates the spaces $W_2^r(R^n)$ are mapped into themselves, an analogous assertion is valid for any sufficiently smooth manifolds of dimension j. In particular, it is valid for any algebraic manifold of dimension j, since every algebraic manifold can be represented as a union of a finite number of locally regular manifolds.

We can now formulate and prove the uniqueness theorem for the solution.

Theorem 1. Suppose we are given a system of differential equations with constant coefficients

$$\sum_{k=1}^{m} P_{kl}\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right) u_k(x_1, \dots, x_n) = 0. \tag{2}$$

Let N be the algebraic manifold of real $(\sigma_1, \dots, \sigma_n)$ for which the algebraic system

$$\sum_{k=1}^{m} P_{kl}(\sigma_1, \dots, \sigma_n) y_k = 0$$

has a nontrivial solution. Let j be the dimension of the manifold N. Under these assumptions, every solution of the system (2) satisfying the condition

$$\sum_{k=1}^{m} \int |u_k(x_1, \dots, x_n)|^2 (1 + |x|)^{j-n} \, dx_1 \cdots dx_n < \infty \tag{3}$$

is identically zero.

Proof. Let $\hat{u}_k(\sigma_1, \dots, \sigma_n)$ be the Fourier transform of $u_k(x_1, \dots, x_n)$, taken in the sense of generalized functions. It follows from (2) that

$$\sum_{k=1}^{m} P_{kl}(\sigma_1, \dots, \sigma_n) \hat{u}(\sigma_1, \dots, \sigma_n) = 0,$$

whence it follows that the supports of $\hat{u}_k(\sigma_1, \dots, \sigma_n)$ are located in N. On the other hand, from (3) we obtain that $\hat{u}_k(\sigma_1, \dots, \sigma_n)$ is a functional on the space $W_2^{\frac{n-j}{2}}(R^n)$, whose support is located in N, whence it follows that $\hat{u}_k(\sigma_1, \dots, \sigma_n) \equiv 0$, since N is a thin set with respect to $W_2^{\frac{n-j}{2}}(R^n)$.

Remark. By slightly modifying the proof of Theorem 1, one can show that the system (2) has only the trivial solution in the class of all (generalized) functions that are combinations of derivatives of functions with convergent integrals of type (3).

3. We now describe the problem in the whole space for equations satisfying the following assumptions:

a) $P(\sigma_1, \dots, \sigma_n)$ is a hypoelliptic polynomial. This means that all complex zeros of the equation $P(\xi) = 0$, $\xi = \xi + i\eta$, satisfy the inequality $|\eta| \geq a|\xi|^d - b$ with some $1 \geq d > 0$, $a > 0$ (for such equations see ([1],[3])).

b) For all real $\sigma = (\sigma_1, \dots, \sigma_n)$ for which $P(\sigma) \neq 0$, the condition $\operatorname{grad} P(\sigma) \neq 0$ holds.

c) For all real σ for which $P(\sigma) = 0$, the complex vector $\operatorname{grad} P(\sigma)$ is not parallel to any real vector.

From b) and c) it follows that the real zeros of the polynomial $P(\sigma)$ fill several smooth manifolds of dimension $n - 2$.

For example, the equation

$$\Delta u + k^2 u + \frac{\partial}{\partial x_1} u = f$$

satisfies these conditions.

Theorem 2. *If the conditions a), b), and c) are fulfilled for the polynomial $P(\sigma)$, then for any finite function $f(x) \in L_2$ there exists a unique solution of equation (1), locally belonging to L_2, such that*

$$\int |u(x)|^2 (1 + |x|)^{-2r} \, dx < \infty, \tag{4}$$

where r is some (fixed) number, $0 < r \leqslant 1$.

Proof. The uniqueness of such a solution follows from Theorem 1. We shall establish existence after constructing a fundamental solution of equation (1). Since it follows from b) and c) that $1/P(\sigma)$ is a locally summable function, we may take as the fundamental solution the Fourier transform (in the sense of generalized functions) of $1/P(\sigma)$. Let there be a finite number of infinitely differentiable functions $\varphi_j(\sigma)$ such that the support of each function $\varphi_j(\sigma)$ is contained in a neighborhood $|\sigma - \sigma_j| < \varepsilon$, and $\varphi(\sigma) = \sum_j \varphi_j(\sigma)$ is equal to one in the ball of radius $2(b/a)^{1/d}$. Then we may write

$$\frac{1}{P(\sigma)} = \frac{1 - \varphi(\sigma)}{P(\sigma)} + \sum_j \frac{\varphi_j(\sigma)}{P(\sigma)}. \tag{5}$$

Consider some term $\varphi_j(\sigma)/P(\sigma)$ and show that it belongs to the space W_2^r for $r < 0$. If $P(\sigma) \neq 0$ within the support of the function $\varphi_j(\sigma)$, then this term is an infinitely differentiable function. If $P(\sigma)$ vanishes for $|\sigma - \sigma_j| \leqslant \varepsilon$, then, by virtue of properties b) and c), one can make such a change of coordinates that this term in the new coordinates will have the form

$$\frac{\varphi_j(\xi_1, \dots, \xi_n)}{\xi_1 + i\xi_2} = \varphi_j(\xi_1, \dots, \xi_n) \frac{f_j(\xi_3, \dots, \xi_n)}{\xi_1 + i\xi_2}, \tag{6}$$

where $f_j(\xi_3, \dots, \xi_n)$ is a finite infinitely differentiable function equal to one for those $(\xi_1, \dots, \xi_n)$ for which $\varphi_j(\xi_1, \dots, \xi_n) \neq 0$.

We shall show that expression (6) belongs to W_2^r for $r < 0$. For this purpose consider its Fourier transform

$$\psi_j(x) = \varphi_j(\widehat{\xi_1, \dots}, \xi_n) * \frac{f_j(\widehat{\xi_3, \dots}, \xi_n)}{\xi_1 + i\xi_2}. \tag{7}$$

Using the fact that the Fourier transform of $(\xi_1 + i\xi_2)^{-1}$ is $-i(x_1 + ix_2)^{-1}$ (see (7), p. 116), we obtain that expression (7) is an ordinary function such that

$$\int |\psi_j(x)|^2 (1 + |x|)^{2r} \, dx < \infty \tag{8}$$

for any $r < 0$. But (8) means precisely that expression (6) belongs to W_2^r for $r < 0$. Therefore each term in (5), except the first, also belongs to W_2^r for $r < 0$. Further, it is not difficult to show, using the hypoellipticity of the polynomial $P(\sigma)$, that the Fourier transform of $[1 - \varphi(\sigma)]/P(\sigma)$ decreases at infinity faster than any power. After this we obtain that the fundamental solution $\mathcal{E}(x) = 1\widehat{/P(\sigma)}$ for $x \neq 0$ is an ordinary function such that

$$\int_{|x| \geqslant 1} |\mathcal{E}(x)|^2 (1 + |x|)^{-2r} \, dx < \infty.$$

for any $r > 0$. Now we can define the required solution of equation (1) by the formula $u(x) = \mathcal{E}(x) * f(x)$.

The following theorem is proved somewhat more difficultly.

Theorem 3. If for the polynomial $P(\sigma)$ conditions b) and c) are satisfied, then for any finite $f(x) \in D'$ there exists, and moreover is unique, a solution of equation (1) which is a finite linear combination of derivatives of functions with convergent integral (4). The same holds for any functions with convergent integral

$$\int |f(x)|(1 + |x|)\, dx < \infty \tag{9}$$

and for any derivatives of such functions. Moreover, one can indicate a number $m > 0$ such that, if the function $f(x)$ has derivatives up to order m, and integrals of type (9) converge also for these derivatives, then the solution $u(x)$ will be an ordinary function with convergent integral (4).

All such right-hand sides form a normed space. If $f(x)$ is small in the norm of this space, then the integral (4) will also be small.

4. In conclusion we describe the problem in the whole space for a somewhat different class of equations.

Theorem 4. Let the polynomial $P(\sigma)$ have only a finite number of isolated real zeros. If $1/P_i(\sigma)$ is a locally summable function, then equation (1), for any right-hand side $f(x)$ which is a combination of summable functions and their derivatives, has a unique solution in the class of functions tending to zero at infinity, and of derivatives of such functions.

The proof of this assertion is comparatively simple, and we shall not give it here. We note only that the conditions of Theorem 4 are satisfied for any homogeneous elliptic equations of order less than the dimension of the space. Along with this, the conditions of Theorem 4 are also satisfied for the heat equation. A large class of equations satisfying these conditions can be distinguished with the aid of the following lemma (originally this lemma was proved by B. R. Vainberg for the case of two variables).

Lemma. Let the polynomial $P(\sigma)$ have an isolated zero at the origin. If $\operatorname{grad} P(\sigma)|_{\sigma=0} \neq 0$, then in some neighborhood of the origin $1/P(\sigma)$ is a summable function.

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CITED LITERATURE

1. G. E. Shilov, *UMN*, **14**, no. 5 (89), 3 (1959).

2. L. Hörmander, L. Liops, *Math. Scand.*, **4**, 259 (1956).

3. L. Hörmander, On the theory of general differential operators in partial derivatives, IL, 1959.

4. V. P. Palamodov, *DAN*, **129**, no. 4, 740 (1959).

5. V. P. Palamodov, *DAN*, **132**, no. 3, 528 (1960).

6. S. M. Nikol' skii, *UMN*, **16**, no. 5 (101), 63 (1961).

7. L. Schwartz, *Théorie des distributions*, 1, 2, Paris, 1950.

Note: Figure translations are in progress. See original paper for figures.

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