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1962

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Abstract

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ON SOME ESTIMATES OF DIFFERENTIABLE FUNCTIONS OF ONE VARIABLE

(Presented by Academician S. L. Sobolev on 15 I 1962)

In interpolation theory the following estimate is well known: if an n-times differentiable function $x(t)$ has n zeros on $[a, b]$,

$$x(a_1) = x(a_2) = \dots = x(a_n) = 0 \quad (a \le a_1 \le \dots \le a_n \le b), \tag{1}$$

then

$$\left|x^{(k)}(t)\right| \le \frac{1}{(n-k)!}\,\mu(b-a)^{n-k}, \quad k = 0, 1, \dots, n-1, \quad a \le t \le b, \tag{2}$$

where

$$\mu = \max_{a<t<b} \left|x^{(n)}(t)\right|. \tag{3}$$

Inequalities (2) are usually used to estimate the remainder term of the Lagrange interpolation formula and its derivatives. The constants $\frac{1}{(n-k)!}$ are, obviously, sharp. However, if one restricts consideration to the interval $[a_1, a_n]$ (which corresponds to interpolation in the exact sense of the word), then the estimates (2) can be improved.

Theorem 1. *If the relations (1), (3) are satisfied, then the estimates are valid*

$$(k = 0, 1, \dots, n-1)$$

$$\left|x^{(k)}(t)\right| \le C_{n,k}\, m\, (a_n - a_1)^{n-k} \quad (a_1 \le t \le a_n), \tag{4}$$

where it is put

$$m = \max_{a_1 \le t \le a_n} \left|x^{(n)}(t)\right|, \tag{5}$$

$$C_{n,0} = \frac{(n-1)^{n-1}}{n!\, n^n}, \quad C_{n,k} = \frac{k}{(n-k)!\, n} \quad (k = 1, 2, \dots, n-1). \tag{6}$$

The indicated values of the constants $C_{n,k}$ $(k = 0, 1, \dots, n-1)$ are unimprovable, since they are attained on the functions

$$x_1(t) = (t - a_1)^{n-1}(a_n - t), \quad x_2(t) = (t - a_1)(a_n - t)^{n-1}$$

(and only on these functions, up to a constant factor). The estimate for $|x(t)|$, i.e., the determination of $C_{n,0}$, is carried out elementarily; the main difficulty of the theorem consists in obtaining estimates for the derivatives. The authors know two different proofs of Theorem 1. One of them uses, in particular, the theorem of M. G. Krein–D. P. Milman ([1]) on extremal points; the second is based on the following representation for $x(t)$, satisfying condition (1):

$$x(t) = (a_n - t)(a_{n-1} - t)\cdots(a_2 - t)\int_{a_1}^{t}\frac{1}{(a_2 - t_1)^2}\,dt_1\int_{a_2}^{t_1}\frac{(a_2 - t_2)}{(a_3 - t_2)^3}\,dt_2\cdots$$
$$\cdots\int_{a_{n-1}}^{t_{n-2}}\frac{(a_{n-1} - t_{n-1})^{n-2}}{(a - t_{n-1})^n}\,dt_{n-1}\int_{a_n}^{t_{n-1}}(a_n - t_n)^{n-1}x^{(n)}(t_n)\,dt_n. \quad (7)$$

Besides questions of interpolation, Theorem 1 finds application in the theory of multipoint boundary-value problems for ordinary differential equations (which in fact served as the starting point of the investigation).

Let, for the equation

$$x^n = f(t, x, x', \dots, x^{n-1}) \qquad (a_1 \le t \le a_m) \quad (8)$$

the problem be considered of finding a solution satisfying the conditions

$$x(a_k) = A_{1,k}, \qquad x'(a_k) = A_{2,k}, \dots, x^{(r_k-1)}(a_k) = A_{r_k,k}, \quad (9)$$

$$k = 1, 2, \dots, m, \quad a_1 < a_2 < \cdots < a_m, \quad r_1 + r_2 + \cdots + r_m = n, \quad 2 \le m \le n.$$

Theorem 2. Let, for $a_1 \le \vartheta \le a_m$, the function $f(\vartheta, \vartheta_0, \dots, \vartheta_{n-1})$ be continuous in ϑ and satisfy the Lipschitz condition in the variables $\vartheta_0, \dots, \vartheta_{n-1}$ with constants $L_0, \dots, L_{n-1}$, respectively:

$$|f(\vartheta, \vartheta_0, \dots, \vartheta_{n-1}) - f(\vartheta, \overline{\vartheta}_0, \dots, \overline{\vartheta}_{n-1})| \le \sum_{i=0}^{n-1} L_i|\vartheta_i - \overline{\vartheta}_i| \qquad (a_1 \le \vartheta \le a_m). \quad (10)$$

If the inequality

$$\sum_{k=0}^{n-1} C_{n,k} L_k (a_m - a_1)^{n-k} < 1 \tag{11}$$

is satisfied, then the boundary-value problem (8)–(9), for arbitrary $A_{i,k}$, has a unique solution.

Proof. Problem (8)–(9) is equivalent to the integral equation

$$x(t) = P(t) + \int_{a_1}^{a_m} G(t,s) f(s, x(s), \dots, x^{(n-1)}(s))\, ds, \tag{12}$$

where $P(t)$ is a polynomial of degree $n-1$ satisfying conditions (9), and $G(t,s)$ is the Green function of the boundary-value problem

$$x^{(n)} = h(t); \tag{13}$$

$$x^{(i)}(a_k) = 0 \qquad (k = 1, 2, \dots, m;\ i = 0, 1, \dots, r_k - 1). \tag{14}$$

Let $M = M(a_1, \dots, a_m;\ A_{1,1}, \dots, A_{r_m,m})$ be the set of functions that are n-times continuously differentiable on $[a_1, a_m]$ and satisfy conditions (9). Define on M the operator B by the formula

$$Bx = P(t) + \int_{a_1}^{a_m} G(t,s) f(s, x(s), \dots, x^{(n-1)}(s))\, ds. \tag{15}$$

This operator obviously maps M into itself. If on the set M we introduce the metric

$$\rho(x,y) = \max_{a_1 \le t \le a_m} |x^{(n)}(t) - y^{(n)}(t)| \qquad (x, y \in M),$$

then M becomes a complete metric space, and the operator B will be a contraction operator on M in this metric. Indeed, taking into account that $x(t) - y(t)$ has on $[a_1, a_m]$ no fewer than n zeros, we find ⋯

$$\rho(Bx, By) = \max_{a_1 \le t \le a_m} \left| f(t, x, \dots, x^{(n-1)}) - f(t, y, \dots, y^{(n-1)}) \right| \le$$

$$\le \max_{a_1 \le t \le a_m} \sum_{k=0}^{n-1} L_k \left| x^{(k)}(t) - y^{(k)}(t) \right| \le$$

$$\leq \max_{a_1 \leq t \leq a_m} \left| x^{(n)}(t) - y^{(n)}(t) \right| \sum_{k=0}^{n-1} L_k C_{n,k} (a_n - a_1)^{n-k} = q\rho(x, y),$$

where by q we have denoted the left-hand side of (11). The operator B therefore has in M a single fixed point, which is equivalent to the existence and uniqueness of the solution of problem (8)–(9). The theorem is proved.

The fact that B is a contraction operator in M means, among other things, that under the conditions of Theorem 2 the method of successive approximations can be applied to find the solution of problem (8)–(9). The rate of convergence to the solution will then be no slower than a geometric progression with ratio q.

If it is assumed that (10) holds only in some part R of the n-dimensional space of values $\vartheta_0, \vartheta_1, \dots, \vartheta_{n-1}$, then one can assert only that, under condition (11), problem (8)–(9) has no more than one solution lying in the part R of the phase plane $(x, x', \dots, x^{(n-1)})$. This assertion is a strengthening of an analogous theorem of Vallée-Poussin ([2]) (see also ([3])).

If one restricts oneself to the existence of a solution, then instead of the Lipschitz condition (10) it is sufficient to require a restriction on the growth of the function $f(\vartheta, \vartheta_0, \dots, \vartheta_{n-1})$.

Theorem 3. Let $f(\vartheta, \vartheta_0, \dots, \vartheta_{n-1})$ be continuous and satisfy the condition

$$|f(\vartheta, \vartheta_0, \dots, \vartheta_{n-1})| \leq L + L_0|\vartheta_0| + L_1|\vartheta_1| + \dots + L^{n+1}|\vartheta_{n-1}| \quad (a_1 \leq \vartheta \leq a_n), \tag{16}$$

where L is any number, and $L_0, \dots, L_{n-1}$ satisfy inequality (11). Then problem (8)–(9) has at least one solution.

Proof. Put

$$l = \max_{a_1 \leq t \leq a_m} \sum_{k=0}^{n-1} L_k \left| P^{(k)}(t) \right|.$$

We shall show that the operator B maps into itself the ball of radius $\dfrac{L+l}{1-q}$ of the space M. Indeed, if $y \in M$ and $\rho(y, P) \leq \dfrac{L+l}{1-q}$, then

$$\rho(By, P) = \max_{a_1 \leq t \leq a_m} \left| f(t, y, \dots, y^{(n-1)}) \right| \leq L + \max_{a_1 \leq t \leq a_m} \sum_{k=0}^{n-1} L_k \left| y^{(k)} \right| =$$

$$= L + \max_{a_1 \leq t \leq a_m} \sum_{k=0}^{n-1} L_k \left| (y-P)^{(k)} + P^{(k)} \right| \leq L + l + \max_{a_1 \leq t \leq a_m} \sum_{k=0}^{n-1} L_k \left| (y-P)^{(k)} \right| \leq$$

$$\leq L + l + q \max_{a_1 \leq t \leq a_m} \left|y^{(n)}\right| \leq L + l + q\frac{L+l}{1-q} = \frac{L+l}{1-q}.$$

In view of the continuity of $f(\vartheta, \vartheta_0, \dots, \vartheta_{n-1})$, the operator B is completely continuous in M; therefore, by Schauder' s principle, B has at least

at least one fixed point. Thus problem (8)–(9) has at least one solution $x(t)$ satisfying the condition

$$\left|x^{(n)}(t)\right| \leqslant \frac{L+l}{1-q} \qquad (a_1 \leqslant t \leqslant a_m).$$

Hence, by virtue of Theorem 1, the a priori estimates follow:

$$\left|x^{(k)}(t) - P^{(k)}(t)\right| \leqslant C_{n,k}\frac{L+l}{1-q}(a_m - a_1)^{n-k},$$

$$k = 0, 1, \dots, n-1 \qquad (a_1 \leqslant t \leqslant a_m).$$

The authors express their gratitude to M. A. Krasnosel' skii, who, in particular, drew their attention to Theorem 3.

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Received
16 XII 1961

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Note: Figure translations are in progress. See original paper for figures.