DIRECT AND INVERSE THEOREMS IN THE THEORY OF APPROXIMATION OF FUNCTIONS BY m-SINGULAR INTEGRALS

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Abstract

Full Text

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DIRECT AND INVERSE THEOREMS IN THE THEORY OF APPROXIMATION OF FUNCTIONS BY m-SINGULAR INTEGRALS

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Let $K_\lambda(t)$ be an even function defined on $(-\infty, \infty)$ and depending on a real parameter λ. Consider the m-singular integral

$$T_\lambda^{[m]}(f;x) = (-1)^{m+1} \int_{-\infty}^{\infty} \left[\sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt) \right] K_\lambda(t)\, dt \qquad (1)$$

for each function $f(t)$ $(-\infty < t < \infty)$, where $m \geq 1$ is a certain fixed positive integer (see ([5])).

The Fourier transform of a function $f(t) \in L_p(-\infty, \infty)$ $(1 \leq p \leq 2)$ will be denoted by $(F)f(t) = F[f(t)]$.

In addition to the Fourier transform, in what follows we shall also need the Fourier-Stieltjes transform of functions of bounded variation on $(-\infty, \infty)$, i.e., functions from the class $BV(-\infty, \infty)$. The Fourier-Stieltjes transform of a function $h(t) \in BV(-\infty, \infty)$ will be denoted by

$$(FS)h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt}\, dh(t).$$

Let us note that if $f(t) \in L_p(-\infty, \infty)$ $(p \geq 1)$ and $K_\lambda(t) \in L(-\infty, \infty)$, then the m-singular integral (1) exists almost everywhere on $(-\infty, \infty)$ and

$$T_\lambda^{[m]}(f;x) \in L_p(-\infty, \infty) \qquad (p \geq 1).$$

In what follows, everywhere we put

$$\psi(x) = F[f(t)], \qquad \varphi_\lambda(x) = F[K_\lambda(t)].$$

Theorem 1. Let $f(t) \in L_p(-\infty, \infty)$ $(1 \leq p \leq 2)$, $K_\lambda(t) \in L(-\infty, \infty)$, and suppose there exists a nonnegative function $\gamma(\lambda)$, monotonically decreasing to zero as $\lambda \to \infty$, such that

$$\lim_{\lambda\to\infty} \frac{1-\sqrt{2\pi}\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}\varphi_\lambda(kx)}{\gamma(\lambda)} = r_m(x) \equiv r(x) \tag{2}$$

for every real x, where $r(x)$ is some continuous real-valued function, and $r(x) \neq 0$ for $-\infty < x < \infty$.

Then from the relation

$$\left\| T_\lambda^{[m]}(f;x) - f(x) \right\|_{L_p} = o[\gamma(\lambda)] \tag{3}$$

it follows that $f(x) = 0$ almost everywhere on $(-\infty, \infty)$.

Theorem 2. Let $f(t), K_\lambda(t) \in L(-\infty, \infty)$, and suppose condition (2) is satisfied.

If

$$\left\| T_\lambda^{[m]}(f;x) - f(x) \right\|_{L} = O[\gamma(\lambda)] \tag{4}$$

as $\lambda \to \infty$, then the function $r(x)\psi(x)$ is the Fourier-Stieltjes transform of some function $h(t) \in BV(-\infty, \infty)$, i.e.

$$r(x)\psi(x) = (FS)h(t) \tag{5}$$

for every $-\infty < x < \infty$.

If, in particular, $r(x) = x^k$, where $k \geq 1$ is an integer, then the function $f(t)$ has a $(k-1)$-st derivative $f^{(k-1)}(t)$ and $f^{(k-1)}(t) \in BV(-\infty, \infty)$.

Theorem 2 is in a certain sense reversible.

Theorem 3. Let $f(t), K_\lambda \in (t) \in L(-\infty, \infty)$ and suppose that the condition

$$\frac{1-\sqrt{2\pi}\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}\varphi_\lambda(kx)}{\gamma(\lambda)r(x)} \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixt}\, dQ_\lambda(t)\, (FS)Q_\lambda(t), \tag{6}$$

is satisfied, where $Q_\lambda(t) \in BV(-\infty, \infty)$, $[\operatorname{Var} Q_\lambda(t)]_{-\infty}^{\infty} \leq M < +\infty$.

Then from condition (5), where $h(t) \in BV(-\infty, \infty)$, (4) follows as $\lambda \to \infty$.

Now let us additionally suppose that the condition

$$\lim_{\lambda\to\infty} (FS)Q_\lambda(t) = 1 \tag{7}$$

holds.

In this case condition (2) is also satisfied. Consequently, from Theorems 2 and 3 it follows:

Theorem 4. Let $f(t), K_\lambda(t) \in L(-\infty,\infty)$ and let conditions (6) and (7) be satisfied.

Then a necessary and sufficient condition for relation (4) to hold as $\lambda \to \infty$ is (5).

We now give a definition (see ([1,2])). Let $\gamma(\lambda) \geq 0$ be a nonincreasing function and $\lim_{\lambda\to\infty}\gamma(\lambda) = 0$. Suppose that there exists a class E of functions $f(x)$ such that:

1°. From $\|T_\lambda^{[m]}(f;x) - f(x)\| = o[\gamma(\lambda)]$ it follows that $f(x) = \text{const}$.

2°. From $\|T_\lambda^{[m]}(f;x) - f(x)\| = O[\gamma(\lambda)]$ it follows that $f(x) \in E$.

3°. For every function $f(x) \in E$ one has

$$\|T_\lambda^{[m]}(f;x) - f(x)\| = O[\gamma(\lambda)] \quad \text{as } \lambda \to \infty.$$

Then the m-singular integral (1) is said to be **saturated with order** $O[\gamma(\lambda)]$, and E is called the **class of saturation**.

We note that Theorems 1 and 4 determine the class and the order of saturation of the m-singular integrals (1) in the space $L(-\infty,\infty)$. From these theorems it follows that the order of saturation of the m-singular integrals is $\gamma(\lambda)$, and the class of saturation consists of those functions $f(x)$ for which condition (5) is satisfied.

We note that the saturation classes of many known summability methods were found by A. Kh. Turetskii ([3,4]).

Theorem 5. Let $f(t) \in L_p(-\infty,\infty)$ $(1 < p \leq 2)$, $K_\lambda(t) \in L(-\infty,\infty)$, and let condition (2) be satisfied for all real x.

If

$$\|T_\lambda^{[m]}(f;x) - f(x)\|_{L_p} = O[\gamma(\lambda)] \tag{8}$$

as $\lambda \to \infty$, then the function $r(x)\psi(x)$ is the Fourier transform of some function $\mu(t) \in L_p(-\infty,\infty)$ $(1 < p \leq 2)$, i.e. the relation

$$r(x)\psi(x) = F[\mu(t)] \tag{9}$$

holds.

If, in particular, $r(x) = x^k$ or $r(x) = |x|^k$, then

$$f^{(k)}(t) \in L_p(-\infty, \infty).$$

Theorem 5 is converse under a small additional restriction.

Theorem 6. Let $f(t) \in L_p(-\infty, \infty)$ $(1 < p \le 2)$, $K_\lambda(t) \in L(-\infty, \infty)$, and suppose the condition

$$\frac{1 - \sqrt{2\pi} \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \varphi_\lambda(kx)}{\gamma(\lambda) r(x)} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt}\, dQ_\lambda(t) = (FS)Q_\lambda(t), \tag{10}$$

is satisfied, where

$$Q_\lambda(t) = \int_{-\infty}^{t} b_\lambda(u)\, du, \qquad \int_{-\infty}^{\infty} |b_\lambda(u)|\, du \le M < +\infty.$$

Then condition (9), where $\mu(t) \in L_p(-\infty, \infty)$, implies (8) as $\lambda \to \infty$.

From Theorems 5 and 6 it follows:

Theorem 7. Let $f(t) \in L_p(-\infty, \infty)$ $(1 < p \le 2)$, $K_\lambda(t) \in L(-\infty, \infty)$, let condition (10) be satisfied, and

$$\lim_{\lambda \to \infty} (FS)Q_\lambda(t) = 1. \tag{11}$$

Then the necessary and sufficient condition for relation (8) to hold is (9).

Theorems 1 and 7 determine the class and the order of saturation of the m-singular integrals (1) in the space $L_p(-\infty, \infty)$ $(1 < p \le 2)$. More precisely, it follows from the indicated theorems that the order of saturation of the m-singular integrals (1) is $\gamma(\lambda)$, and the saturation class consists of those functions $f(x)$ for which condition (9) is satisfied.

In particular, if $m = 1$ and $K_\lambda(t) = \dfrac{\lambda}{\sqrt{2\pi}} K(\lambda t)$, then from Theorems 1, 2, and 5 the corresponding results of Sunouchi ([1]) follow.

In the case $m = 1$, $r(x) = c|x|^\alpha$ $(c > 0,\ 0 < \alpha \le 2)$, $\gamma(\lambda) = \lambda^{-\alpha}$, and $K_\lambda(t) = \dfrac{\lambda}{\sqrt{2\pi}} K(\lambda t)$, our theorems yield the corresponding results of Butzer ([2]).

It is known ([7]) that for functions $f(t) \in L_p(-\infty, \infty)$ $(2 < p \le \infty)$ the Fourier transform in the ordinary sense, generally speaking, does not exist. However, using the notion of summability of integrals by the Cesàro method, the Fourier transform can be defined ([6]). Sunouchi ([1]) partially used these results to determine the saturation of singular integrals with kernels of Fejér type in the space $L_p(-\infty, \infty)$ $(2 < p \leq \infty)$.

If one uses these definitions of Offord ([6]) and the corresponding schemes of Sunouchi ([1]) and Butzer ([2]), one obtains the corresponding direct and inverse theorems for the approximation of functions by m-singular integrals in the space $L_p(-\infty, \infty)$ $(2 < p \leq \infty)$.

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Note: Figure translations are in progress. See original paper for figures.