



Soviet-era science, translated into English

MATHEMATICS

P. D. KALAFATI

1962

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196201.18472>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

P. D. KALAFATI

OSCILLATION PROPERTIES OF FUNDAMENTAL FUNCTIONS OF THIRD-ORDER BOUNDARY-VALUE PROBLEMS

(Presented by Academician I. G. Petrovskii, 21 XI 1961)

We consider a boundary-value problem consisting of the equation

$$D_3y - \lambda p(x)y = 0, \quad (1)$$

where

$$D_3y = \rho_0(x) \frac{d}{dx} \rho_1(x) \frac{d}{dx} \rho_2(x) \frac{d}{dx} \rho_3(x)y; \quad \rho_i(x) > 0 \quad (i = 0, 1, 2, 3); \quad p(x) > 0$$

$$(0 \leq x \leq 1),$$

and of three linearly independent boundary conditions

$$(\alpha_{i1}D_0y + \alpha_{i2}D_1y + \alpha_{i3}D_2y)|_{x=0} + (\beta_{i1}D_0y + \beta_{i2}D_1y + \beta_{i3}D_2y)|_{x=1} = 0 \quad (2)$$

$$(i = 1, 2, 3),$$

where we put

$$D_0y = \rho_3(x)y(x), \quad D_1y = \rho_2(x) \frac{d}{dx} \rho_3(x)y(x),$$

$$D_2y = \rho_1(x) \frac{d}{dx} \rho_2(x) \frac{d}{dx} \rho_3(x)y(x).$$

We consider the case in which problem (1), (2) has a discrete spectrum. Then, without loss of generality in the reasoning, it may be assumed to be nondegenerate.

Our investigation is reduced to the study of the Green's function of the operator D_3y corresponding to the boundary conditions (2). As is known, problem (1), (2) is equivalent to solving the integral equation

$$y(x) = \lambda \int_0^1 K \left(\begin{matrix} x \\ s \end{matrix} \right) y(s) d\sigma(s), \quad (3)$$

where $K \left(\begin{matrix} x \\ s \end{matrix} \right)$ is the Green's function of the problem under consideration, and $d\sigma(s) = p(s) ds$. We substantially generalize the methods and results of the work ⁽¹⁾ as applied to third-order equations.

Slightly generalizing the generally accepted terminology, we shall call a kernel $K \left(\begin{matrix} x \\ s \end{matrix} \right)$ **oscillatory** if for each n one can choose a number ε_n , equal either to 1 or to -1 , such that the inequalities

$$\varepsilon_n K \left(\begin{matrix} x_1 x_2 \dots x_n \\ s_1 s_2 \dots s_n \end{matrix} \right) \geq 0; \quad (4)$$

$$\varepsilon_n K \left(\begin{matrix} s_1 s_2 \dots s_n \\ s_1 s_2 \dots s_n \end{matrix} \right) > 0, \quad (5)$$

hold if $0 \leq x_1 < x_2 < \dots < x_n \leq 1$ and $0 \leq s_1 < s_2 < \dots < s_n \leq 1$. If, however, inequalities (4) and (5) are valid only for even (odd) values of n , then the kernel is called **even (odd)** oscillatory.

Let us renumber the eigenvalues of the kernel $K \left(\begin{matrix} x \\ s \end{matrix} \right)$ in increasing order of their moduli (in the case of distinct moduli the order is immaterial), counting each such number as many times as its multiplicity, with the first of them ob-

denote it by λ_0 . Then, for an ordinary oscillatory kernel, the eigenvalues are real, distinct in absolute value, and the n -th fundamental function has exactly n zeros. For an even oscillatory kernel

$$|\lambda_0| \leq |\lambda_1| < |\lambda_2| \leq |\lambda_3| < \dots,$$

and for an odd one

$$|\lambda_0| < |\lambda_1| \leq |\lambda_2| < |\lambda_3| \leq \dots.$$

If $K \left(\begin{matrix} x \\ s \end{matrix} \right)$ is an even (odd) oscillatory kernel and the eigenvalue is simple, then the n -th eigenfunction, if it is real, has, for even (odd) n , either n or $n + 1$ zeros, and for odd (even) n , either n or $n - 1$ zeros. If this fundamental function is complex, then the numbers of zeros of its real and imaginary parts lie within the same bounds. Finally, if there are multiple eigenvalues (they are necessarily real), to which one eigenfunction corresponds, then, in order that the preceding formulation remain valid, it is sufficient to augment the eigenfunctions by associated ones ⁽²⁻⁴⁾.

Denoting the determinants of order 3 of the matrix of boundary conditions

$$\left\| \begin{array}{cccccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & \beta_{11} & \beta_{12} & \beta_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_{21} & \beta_{22} & \beta_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \beta_{31} & \beta_{32} & \beta_{33} \end{array} \right\| \quad (6)$$

by the numbers of the columns entering into them, we divide these determinants into 4 groups in the following way: to the 1st group we assign the determinant (123); to the 2nd—the determinants in which two columns consist of α -elements and one of β -elements; to the 3rd—the determinants in which one column consists of α -elements and two of β -elements; to the 4th—the determinant (456).

In the second (or, respectively, in the third) group we shall call the determinant (236) (or, respectively, (356)) senior if it is different from zero. If it is equal to zero, we shall call senior those among the determinants (235) and $-(136)$ (i.e. the determinant (136) with the opposite sign) (or, respectively, (346) and $-(256)$) which are different from zero. If all the listed determinants of the 2nd (respectively, 3rd) group are equal to zero, then we shall call senior those among the determinants (126), $-(135)$, and (234) (or (156), $-(246)$, and (345)) which are different from zero. If these determinants too are equal to zero, then, by definition, the senior ones will be those among the determinants (125) and $-(134)$ (or (146) and $-(245)$) which are different from zero. There remains still the determinant (124) (respectively (145)), which will be senior if all the other determinants of the 2nd (or 3rd) group are equal to zero.

If the signs of any senior determinants of the same group (and there may be two or three of them) are the same, we shall agree to say that the group is **definite**; otherwise we shall say that the group is **indefinite**.

We now pass to the formulation of the results obtained. The following cases are possible:

A. The matrix of boundary conditions can be reduced to one of the forms:

$$A_1. \left\| \begin{array}{cccccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_{31} & \beta_{32} & \beta_{33} \end{array} \right\|, \quad (7_1) \quad A_2. \left\| \begin{array}{cccccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_{21} & \beta_{22} & \beta_{23} \\ 0 & 0 & 0 & \beta_{31} & \beta_{32} & \beta_{33} \end{array} \right\|. \quad (7_2)$$

Then, if each of the symbols

$$K \begin{pmatrix} x_1 \\ s_1 \end{pmatrix}, \quad K \begin{pmatrix} x_1 & x_2 \\ s_1 & s_2 \end{pmatrix}, \quad (0 \leq x_1 \leq x_2 \leq 1, 0 \leq s_1 \leq s_2 \leq 1)$$

does not change sign, the Green's function is an oscillatory kernel. Only in case A can the Green's function be an ordinary oscillatory kernel.

B. The matrix of boundary conditions can be reduced to one of the forms:

$$B_1 \cdot \left\| \begin{array}{cccccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \beta_{31} & \beta_{32} & \beta_{33} \end{array} \right\|, \quad (123) \neq 0; \quad (8_1)$$

$$B_2 \cdot \left\| \begin{array}{cccccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & \beta_{11} & \beta_{12} & \beta_{13} \\ 0 & 0 & 0 & \beta_{21} & \beta_{22} & \beta_{23} \\ 0 & 0 & 0 & \beta_{31} & \beta_{32} & \beta_{33} \end{array} \right\|, \quad (456) \neq 0_4. \quad (8_2)$$

Now, in order that the Green' s function be an even or odd oscillatory kernel, it is necessary and sufficient that the boundary conditions with matrices (7₁) in the case B₁ (or (7₂) in the case B₂), where all elements α_{ik} and β_{ik} are the same as in the matrices (8₁) (or (8₂)), correspond to Green' s functions that are oscillatory kernels. Moreover, in the case B₁ (or B₂) the 2nd (3rd) group is definite, and if the sign of the leading determinant of this group coincides with the sign of the determinant (123) (or (456)), then the corresponding Green' s function will be an even oscillatory kernel, and otherwise an odd one.

C. The matrix of boundary conditions can be reduced to the form

$$\left\| \begin{array}{cccccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_{21} & \beta_{22} & \beta_{23} \\ 0 & 0 & 0 & \beta_{31} & \beta_{32} & \beta_{33} \end{array} \right\|, \quad (*)$$

where not all determinants of order 2 of the form

$$\begin{vmatrix} \alpha_{1i} & \alpha_{1k} \\ \alpha_{2i} & \alpha_{2k} \end{vmatrix}$$

and also not all determinants of order 2 of the form

$$\begin{vmatrix} \beta_{2i} & \beta_{2k} \\ \beta_{3i} & \beta_{3k} \end{vmatrix}$$

are equal to zero. In this case, in order that the corresponding Green' s function be an even or odd oscillatory kernel, it is necessary and sufficient that both auxiliary Green' s functions satisfying the boundary conditions with matrices (7₁) and (7₂), where the elements α_{ik} and β_{ik} are the same as in the matrix (*), be oscillatory kernels. Moreover, the 2nd and 3rd groups of the matrix (*) are definite, and if the signs of the leading determinants of these groups coincide, then the Green' s function for the boundary conditions with matrix (*) will be an even oscillatory kernel, and otherwise an odd one.

To pass to more complicated boundary conditions, we introduce the notion of the symbols

$$K_2 \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ s_1 & s_2 & \dots & s_n \end{bmatrix} \quad \text{and} \quad K_3 \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ s_1 & s_2 & \dots & s_n \end{bmatrix}.$$

Consider any Green' s function of problem (1), (2). Let its Fredholm symbols be

$$K \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ s_1 & s_2 & \dots & s_n \end{pmatrix}.$$

These symbols are uniquely expressed in terms of the Fredholm symbols of the Cauchy function of the operator D_3y and in terms of their derivatives, the coefficients being the determinants of order 3 of the matrix (6). Such a representation is unique and is given in paper (1). In order to find

$$K_2 \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ s_1 & s_2 & \dots & s_n \end{bmatrix}$$

(or

$$K_3 \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ s_1 & s_2 & \dots & s_n \end{bmatrix},$$

) it is sufficient in this expression to replace by zeros the determinants of all groups except the determinants of the 2nd group (3rd group), leaving the determinants of this group unchanged.

Let us now consider the remaining cases.

G. The matrix of boundary conditions can be reduced to one of the forms:

$$\Gamma_1. \quad \left\| \begin{array}{cccccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_{21} & \beta_{22} & \beta_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \beta_{31} & \beta_{32} & \beta_{33} \end{array} \right\|; \quad (9_1)$$

$$\Gamma_2. \quad \left\| \begin{array}{cccccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & \beta_{11} & \beta_{12} & \beta_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_{21} & \beta_{22} & \beta_{23} \\ 0 & 0 & 0 & \beta_{31} & \beta_{32} & \beta_{33} \end{array} \right\|, \quad (9_2)$$

where in the first case (123) $\neq 0$ and not all determinants

$$\begin{vmatrix} \beta_{2i} & \beta_{2k} \\ \beta_{3i} & \beta_{3k} \end{vmatrix}$$

are equal to ze-

...but in the second case (456) $\neq 0$ and not all determinants

$$\begin{vmatrix} \alpha_{1i} & \alpha_{1k} \\ \alpha_{2i} & \alpha_{2k} \end{vmatrix}$$

are equal to zero. Then, in order that the Green' s function be an even or odd oscillation kernel, it is necessary that the auxiliary Green' s function corresponding to the boundary conditions with matrix (7₂) (or, respectively, (7₁)), where the elements α_{ik} and β_{ik} are the same as in the matrices (9₁) (or (9₂)), be an

oscillation kernel, and that the sign of the principal determinant of the 3rd (or 2nd) group coincide with the sign of the determinant (123) (or (456)); and, if these conditions are satisfied, it is sufficient that each of the symbols

$$K_2 \begin{bmatrix} x_1 \\ s_1 \end{bmatrix} \quad \text{and} \quad K \begin{bmatrix} x_1 & x_2 \\ s_1 & s_2 \end{bmatrix}$$

(or

$$K_3 \begin{bmatrix} x_1 \\ s_1 \end{bmatrix} \quad \text{and} \quad K_3 \begin{bmatrix} x_1 & x_2 \\ s_1 & s_2 \end{bmatrix}$$

) not change sign for $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq s_1 \leq s_2 \leq 1$. Then, if the sign of the principal determinant of the 2nd (or 3rd) group coincides with the sign of the determinant (123) (or (456)), the kernel will be an even oscillation kernel, and in the opposite case—an odd oscillation kernel. In this case the 3rd (respectively, the 2nd) group will be definite, and, when the necessary conditions are fulfilled, the 2nd group (respectively, the 3rd) will also be definite.

It remains to analyze the last and most complicated case.

D. The matrix of the coefficients of the boundary conditions has the form (6), where $(123) \neq 0$ and $(456) \neq 0$. In this case, in order that the corresponding Green' s function be an even or odd oscillation kernel, it is necessary that the equalities $\text{sign}(123) = \text{sign}(356)$ and $\text{sign}(236) = \text{sign}(456)$ hold, provided only that $(236) \neq 0$ and $(356) \neq 0$. If some determinant (236) or (356) is equal to zero, then its place in the first case is taken by that one of the determinants (235) and $-(136)$ which is different from zero, and in the second case by that one of the determinants (346) and $-(256)$ which is different from zero. Moreover, if $(235) \neq 0$ and $-(136) \neq 0$, then their signs coincide. The same applies to the determinants (346) and $-(256)$. If both determinants (236) and (356) are equal to zero, then the replacement rules remain in force. It may then be asserted that at least one of the determinants (235) and $-(136)$, and also at least one of the determinants (346) and $-(256)$, is equal to zero. If, finally, all three of the considered determinants of one group are equal to zero (for example, $(236) = (235) = (136) = 0$), then the remaining three determinants are also equal to zero (i.e., $(356) = (346) = (256) = 0$). In this case a necessary condition for oscillation will be

$$\text{sign}(123) = \text{sign}(156) = \text{sign}(135).$$

These equalities imply the equalities

$$\text{sign}(456) = \text{sign}(126) = \text{sign}(234),$$

and conversely. All the indicated determinants can no longer vanish.

When the necessary conditions for even or odd oscillation of the Green's function are satisfied, it is sufficient that each of the symbols

$$K_2 \begin{bmatrix} x_1 \\ s_1 \end{bmatrix}, \quad K_3 \begin{bmatrix} x_1 \\ s_1 \end{bmatrix}, \quad K_2 \begin{bmatrix} x_1 & x_2 \\ s_1 & s_2 \end{bmatrix}, \quad K_3 \begin{bmatrix} x_1 & x_2 \\ s_1 & s_2 \end{bmatrix}$$

not change sign for $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq s_1 \leq s_2 \leq 1$. In this case, if $\text{sign}(123) = \text{sign}(456)$, the Green's function will be an even oscillation kernel, and if $\text{sign}(123) = -\text{sign}(456)$, an odd oscillation kernel.

Odessa Technological Institute
of the Food and Refrigeration Industry

Received
14 XI 1961

REFERENCES

1. P. D. Kalafati, *DAN*, **26**, No. 6, 535 (1940).
2. F. R. Gantmakher, *DAN*, **1** (10), 3-5 (1936).
3. F. R. Gantmakher, M. G. Krein, *Oscillation Matrices and Kernels*, 1950.
4. P. D. Kalafati, *Notes of the Kharkov Mathematical Society*, **25**, ser. 4, 129 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.