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Abstract

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MATHEMATICS

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INVESTIGATION OF SINGULARLY PERTURBED SYSTEMS IN A NEIGHBORHOOD OF A FAMILY OF CLOSED ORBITS

(Presented by Academician N. N. Bogolyubov, 2 I 1962)

In the present note, N. N. Bogolyubov' s method ([1]) for investigating a system of differential equations

$$\frac{dx}{dt} = \varepsilon X(t, x)$$

in standard form is applied to the study of the properties of solutions of the system

$$\frac{dx}{dt} = f(x, z, \varepsilon), \qquad \varepsilon \frac{dz}{dt} = F(x, z, \varepsilon), \tag{1}$$

which is singularly perturbed with respect to the system

$$\frac{d\bar{x}}{dt} = f(\bar{x}, \bar{z}, 0); \qquad F(\bar{x}, \bar{z}, 0) = 0, \tag{2}$$

where x, f and z, F are vectors of dimensions m and n, respectively.

It is assumed that the system $F(x, z, 0) = 0$ has an isolated solution $z = \varphi(x)$, while the unperturbed system

$$\frac{d\bar{x}}{dt} = f(\bar{x}, \varphi(\bar{x})), \qquad \bar{z} = \varphi(\bar{x}) \tag{2'}$$

admits a family of solutions

$$\bar{x} = \bar{x}^0(\theta, c), \qquad \bar{z} = \bar{z}^0(\theta, c), \tag{3}$$

periodic in $\theta = \omega(c)t + \varphi_0$ with period 2π, where c is a vector with components $c_1, c_2, \dots, c_s$, and φ_0 is a scalar $(s + 1 < m)$, with ω in the general case being a function of the vector c, continuous and bounded together with its first-order partial derivatives with respect to $c_1, c_2, \ldots, c_s$ for all $c \in C$; the parameters $\varphi_0, c_1, \ldots, c_s$ are essentially distinct.

With respect to the right-hand sides of system (1) and the vector φ we shall assume that in the domain $x \in U_\rho,\ z \in U_\nu,\ 0 < \varepsilon < \varepsilon^*$, where U_ρ and U_ν denote the ρ- and ν-neighborhoods of the family of orbits, $x^0, \bar{z}^0$, the vector f and its partial derivatives with respect to x and z up to order k, and the vectors F and φ, together with their partial derivatives with respect to x and z up to order $(k+1)$ inclusive, are bounded and uniformly continuous.

We shall prove that system (1) also admits a family of periodic solutions of the form

$$x = x^0(\theta, c, \varepsilon), \qquad z = z^0(\theta, c, \varepsilon)$$

and establish the properties of these solutions.

We transform system (1) by means of the substitution

$$z = \varphi(x) + y \tag{4}$$

to the form

$$\frac{dx}{dt} = \Phi(x, y, \varepsilon), \qquad \varepsilon\frac{dy}{dt} = \mathfrak{A}(x)y + Q(x, y, \varepsilon), \tag{5}$$

where the following notation has been introduced:

$$\Phi(x, y, \varepsilon) = f(x, \varphi(x) + y, \varepsilon), \qquad \mathfrak{A}(x) = \left.\frac{\partial F(x, \varphi(x) + y, \varepsilon)}{\partial z}\right|_{y=0},$$

$$Q(x, y, \varepsilon) = F(x, \varphi(x) + y, \varepsilon) - \frac{\partial F(x, \varphi(x))}{\partial z}\, y - \varepsilon\frac{d\varphi}{dx}\, f(x, \varphi(x) + y, \varepsilon).$$

We note that the functions $\bar{x} = \bar{x}^0(\theta, c)$ and $\bar{y} = 0$ are a solution of the system

$$\frac{d\bar{x}}{dt} = f(\bar{x}, \varphi(\bar{x})), \qquad \varepsilon\frac{d\bar{y}}{dt} = \mathfrak{A}(\bar{x})\bar{y}, \tag{6}$$

for which the variational equations corresponding to the family (3) have the form

$$\frac{d\delta\bar{x}}{dt} = \bar{f}(\bar{x}^0)\delta\bar{x}, \tag{7}$$

$$\varepsilon \frac{d\delta \bar{y}}{dt} = \mathfrak{A}(\bar{x}^0)\delta \bar{y}, \tag{8}$$

where $\bar{f} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial z}\dfrac{d\varphi}{dx}$, and $\mathfrak{A}$ are periodic functions of θ with period 2π.

We shall assume that all nonzero characteristic exponents $a_{s+2}, \dots, a_m$ of equation (7) and all n characteristic exponents of equation (8) have negative real parts. The first $s+1$ characteristic exponents of equation (7) are equal to zero, since $\partial \bar{x}^0/\partial\theta$ and $\partial \bar{x}^0/\partial c_j$ $(j = 1, \dots, s)$ are solutions of this equation.

According to A. M. Lyapunov' s theorem ([2]) and the remarks made in ([3]), there exist real nonsingular matrices $P(\theta, c)$ and $R(\theta, c)$, periodic in θ with period 2π (if none of the real characteristic numbers of equations (7) and (8) is negative∗), possessing continuous first derivatives with respect to θ and c for all $\theta \in \Theta,\ c \in C$, as well as such square real matrices $H(c)$ and $H_1(c)$, that the transformation

$$\delta \bar{x} = \frac{\partial \bar{x}^0}{\partial \theta} u + \frac{\partial \bar{x}^0}{\partial c} v + P(\theta, c) w, \tag{9}$$

$$\delta \bar{y} = R(\theta, c)\xi \tag{10}$$

reduces systems (7) and (8) to equivalent systems with constant coefficients

$$\frac{du}{dt} = 0, \qquad \frac{dv}{dt} = 0, \qquad \frac{dw}{dt} = H(c)w, \tag{11}$$

$$\varepsilon \frac{d\xi}{dt} = H_1(c)\xi, \tag{12}$$

where the eigenvalues of the matrices $H(c)$ and $H_1(c)$ are the nonzero characteristic exponents of equation (7) and the characteristic exponents of equation (8), respectively.

Applying now to system (5) the transformation

$$x = \bar{x}^0(\theta, c) + P(\theta, c)h, \qquad y = R(\theta, c)\xi, \qquad h = (h_{s+2}, \dots, h_m), \tag{13}$$

we obtain the system

$$\begin{aligned} \frac{d\theta}{dt} &= \omega(c) + A_1(\theta, c, h, \xi, \varepsilon), \\ \frac{dc}{dt} &= A_2(\theta, c, h, \xi, \varepsilon), \\ \frac{dh}{dt} &= H(c)h + A_3(\theta, c, h, \xi, \varepsilon), \\ \varepsilon\frac{d\xi}{dt} &= H_1(c)\xi + A_4(\theta, c, h, \xi, \varepsilon). \end{aligned} \tag{14}$$

* Matrices $P(\theta, c)$ and $R(\theta, c)$ will have period 4π in θ if among the real characteristic numbers of equations (

The functions A_i $(i = 1, \dots, 4)$ are defined in the domain

$$\theta \in \Theta, \quad c \in C, \quad h \in U_\rho, \quad \xi \in U_\nu, \quad 0 < \varepsilon < \varepsilon^*, \tag{15}$$

where U_ρ and U_ν denote the ρ- and ν-neighborhoods of the points $h = 0$ and $\xi = 0$, have bounded and uniformly continuous derivatives with respect to θ and c, are periodic in θ, and satisfy the inequalities

$$\begin{gathered} |A_i(\theta, c, 0, 0, \varepsilon)| \le M(\varepsilon), \\ |A_i(\theta', c', h', \xi', \varepsilon) - A_i(\theta'', c'', h'', \xi'', \varepsilon)| \le \\ \le \lambda(\varepsilon, \sigma, \mu)\left(|\theta' - \theta''| + |c' - c''| + |h' - h''| + |\xi' - \xi''|\right), \end{gathered} \tag{16}$$

where $\sigma < \rho,\ \mu < \nu$, and moreover $M(\varepsilon) \to 0,\ \lambda(\varepsilon, \sigma, \mu) \to 0$ together with ε.

The following assertions hold:

1. One can specify a positive number ε_1 such that, for every positive $\varepsilon < \varepsilon_1$, the system of equations (14) has a unique integral manifold, periodic in θ with period 2π, representable by relations of the form

$$h = g(\theta, c, \varepsilon), \qquad \xi = g_1(\theta, c, \varepsilon),$$

where the functions g and g_1 are defined in the domain $\theta \in \Theta,\ c \in C$ and satisfy the inequalities

$$|g(\theta, c, \varepsilon)| \le D(\varepsilon), \qquad |g(\theta', c', \varepsilon) - g(\theta'', c'', \varepsilon)| \le \Delta(\varepsilon)(|\theta' - \theta''| + |c' - c''|), \tag{17}$$

$$|g_1(\theta, c, \varepsilon)| \le D(\varepsilon), \qquad |g_1(\theta', c', \varepsilon) - g_1(\theta'', c'', \varepsilon)| \le \Delta(\varepsilon)(|\theta' - \theta''| + |c' - c''|),$$

with $D(\varepsilon) \to 0,\ \Delta(\varepsilon) \to 0$ together with ε. Moreover, g and g_1 will have bounded and uniformly continuous derivatives up to order k, inclusive.

2. One can specify positive constants $\varepsilon_0, \gamma_0, C_0, \sigma_0, \gamma_1, C_1, \sigma_1$ ($\sigma_0 < \rho,\ \sigma_1 < \nu,\ \varepsilon_0 < \varepsilon_1$) such that, for all $\varepsilon < \varepsilon_0$, any real t_0, and any $\theta \in \Theta,\ c \in C$, there exist an $[m-(s+1)]$-dimensional domain U_{σ_0} of points $\{h\}$ and an n-dimensional domain U_{σ_1} of points ξ with the properties

$$|h_t^* - g(\theta_t, c_t, \varepsilon)| \leq C_0 e^{-\gamma_0(t-t_0)} |h_0^* - g(\theta_0, c_0, \varepsilon)|,$$

$$|\xi_t^* - g_1(\theta_t, c_t, \varepsilon)| \leq C_1 e^{-\gamma_1(t-t_0)} |\xi_0^* - g_1(\theta_0, c_0, \varepsilon)|,$$

where h_t^* and ξ_t^* are arbitrary solutions of system (14); $h_0^* = h_t^*(t_0)$, $\xi_0^* = \xi_t^*(t_0)$, $\theta_0 = \theta_t(t_0)$, $c_0 = c_t(t_0)$.

Taking into account that system (14) is equivalent to system (1), we have the following theorem:

Theorem. If the assumptions made above are satisfied, then one can specify a number $\varepsilon_0 > 0$ such that, for any positive $\varepsilon < \varepsilon_0$, the following assertions hold:

1. System (1) has a unique integral manifold $\mathfrak{M}$.
2. This manifold admits the parametric representation

$$x = \bar{x}^0(\theta, c) + P(\theta, c) g(\theta, c, \varepsilon),$$

$$z = \varphi(x) + R(\theta, c) g_1(\theta, c, \varepsilon),$$

where the right-hand sides are defined in the domain $\theta \in \Theta,\ c \in C,\ 0 < \varepsilon < \varepsilon_0$, are periodic in the angular variable θ with period 2π, and have bounded and uniformly continuous derivatives with respect to θ and c up to order k, inclusive.

3. One can find a function $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$|x(\theta, c, \varepsilon) - \bar{x}^0(\theta, c)| < \delta(\varepsilon), \qquad |z(\theta, c, \varepsilon) - \bar{z}^0(\theta, c)| < \delta(\varepsilon).$$

4. On the manifold $\mathfrak{M}$, system (1) is equivalent to the system

$$\frac{d\theta}{dt} = \omega(c) + A_1^*(\theta, c, \varepsilon), \qquad \frac{dc}{dt} = A_2^*(\theta, c, \varepsilon),$$

where

$$A_1^* = A_1(\theta, c, g(\theta, c, \varepsilon), g_1(\theta, c, \varepsilon), \varepsilon), \quad A_2^* = A_2(\theta, c, g(\theta, c, \varepsilon), g_1(\theta, c, \varepsilon), \varepsilon)$$

are definite functions, periodic in θ with period 2π and possessing bounded and uniformly continuous derivatives with respect to θ and c up to and including order $k-20$.

5. The manifold $\mathfrak{M}$ has the property of attracting solutions of system (1) that are close to it.

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CITED LITERATURE

1. N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 1955.

2. A. M. Lyapunov, *The General Problem of the Stability of Motion*, 1935.

3. J. K. Hale, A. P. Stokes, *Arch. f. Rat. Mech. and Analysis*, 6, No. 2, 133 (1960).

Note: Figure translations are in progress. See original paper for figures.