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F. A. BEREZIN

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Abstract

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F. A. BEREZIN

ON THE LEE MODEL

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1. This paper discusses the possibility of constructing the Lee model within the framework of the Hamiltonian formalism. Apparently it is generally recognized that the Hamiltonian $H_0 + gV$, formally written in terms of creation and annihilation operators, is not an operator in Hilbert space for any field theory. In the present paper a method is proposed for associating with this Hamiltonian a certain regularized operator H_r.

The operator H_r is constructed by means of the following consideration: the operator H_r must be self-adjoint and coincide with H_0 on all those elements f of occupation-number space which satisfy the condition $Vf = 0$. Let us note that the original expression $H_0 + gV$ has the latter property. Denote by $D_{\widetilde{H}_0}$ the above-described set of elements of occupation-number space. The operator H_0, considered only on $D_{\widetilde{H}_0}$, will be denoted by $\widetilde{H}_0$. It is symmetric; H_r is its self-adjoint extension. The construction of self-adjoint extensions of symmetric operators is well studied (see, for example, ([1,2])).

In the Lee model it turns out that $H_r = H_0$. This circumstance is a manifestation of the zero-charge paradox; moreover, it apparently is the mathematical content of the paradox. Clearly, the equality $H_r = H_0$ excludes the possibility of constructing a theory within the Hamiltonian formalism. It turns out, however, that if instead of the ordinary Lee model one considers the two-dimensional one, then nontrivial operators H_r exist for it. It may be verified that a perturbation calculation by means of the operator H_r in the case of the two-dimensional model is completely equivalent to the usual renormalization theory.

Thus it turns out that for the Lee model the zero-charge paradox owes its origin to the dimension of configuration space. Apparently, the same situation also holds in relativistically invariant theories.

It further turns out that if the Hilbert space of occupation numbers is enlarged by introducing an indefinite metric, then in the newly obtained space one can construct a nontrivial operator H_r for the three-dimensional Lee model. A calculation by means of this operator proves to be equivalent to the usual renormalization theory. In this case the appearance of the unphysical pole is explained by general properties of operators in Hilbert spaces with an indefinite scalar product.

Finally, let us note that the Hilbert space with an indefinite scalar product described here differs from that used by Lee, Chew, Pauli, and Heisenberg ($^{3-5}$). The theory of extensions of symmetric operators in this space is in many respects analogous to the usual one.

2. Let us construct the operator H_r in the sector $\begin{pmatrix} V \\ N+\theta \end{pmatrix}$ of the Lee model. Denote, as usual, the mass of the V-particle by m, the coupling constant by g, and the cutoff factor by χ. We denote by $H(m, g, \chi)$ the Hamiltonian corresponding to the cutoff factor χ. As is known, in the sector $\begin{pmatrix} V \\ N+\theta \end{pmatrix}$ the operator

$H(m,\ g,\ \chi)$ is equal to

$$H(m,g,\chi)f = H_0 f + gVf = \begin{pmatrix} mf_0 \\ \omega(k)f_1(k) \end{pmatrix} + g\begin{pmatrix} \displaystyle\int f_1(k)\frac{\chi(k)}{\sqrt{\omega(k)}}\,d^3k \\ \displaystyle f_0\frac{\chi(k)}{\sqrt{\omega(k)}} \end{pmatrix}. \qquad (1)$$

In this formula

$$f = \begin{pmatrix} f_0 \\ f_1(k) \end{pmatrix}; \qquad (f,f) = |f_0|^2 + \int |f_1(k)|^2\,d^3k;$$

$$\chi(k) = \begin{cases} 1 & \text{for } |k| < N, \\ 0 & \text{for } |k| > N; \end{cases} \qquad \omega(k) = \sqrt{k^2+\mu^2}.$$

$H(m,g,\chi)$ is a self-adjoint operator; however, the limiting expression $H(m,g,1)$ is not an operator at all. Indeed, otherwise there would exist elements f of Hilbert space such that $H(m,g,1)f$ is likewise an element of Hilbert space. In particular, in this case the function

$$\varphi(k) = \omega(k)f_1(k) + gf_0\frac{1}{\sqrt{\omega(k)}}$$

would have to have a summable square. Dividing both sides of the equality by $\omega(k)$, we find:

$$gf_0\frac{1}{\omega(k)\sqrt{\omega(k)}} = \frac{\varphi(k)}{\sqrt{\omega(k)}} - f_1(k). \qquad (2)$$

On the right-hand side of the equality there stand functions with summable square; hence the same kind of function stands on the left-hand side. This, however, is possible only if $gf_0 = 0$, since

$$\int \frac{d^3k}{\omega^3(k)} = 4\pi \int_0^\infty \frac{k^2\,dk}{(k^2+\mu^2)^{3/2}} = \infty. \tag{3}$$

Let us note that at this point the dependence on the dimensionality of space appears: in the two-dimensional model, the integral (3) is replaced by

$$2\pi \int_0^\infty \frac{k\,dk}{(k^2+\mu^2)^{3/2}} < \infty.$$

We proceed to the construction of the operator H_r. The set $D_{\widetilde{H}_0}$, as is seen from (1), consists of vectors of the form

$$f_1 = \begin{pmatrix} 0 \\ f_1(k) \end{pmatrix}, \qquad \int \frac{f_1(k)}{\sqrt{\omega(k)}}\,d^3k = 0.$$

In accordance with the general theory of extensions, the operator H_r is defined on the set $D_{H_r} \supset D_{\widetilde{H}_0}$ and acts according to the formula

$$H_r f = \begin{pmatrix} \varkappa \\ \omega(k) f_1(k) + \dfrac{\sigma}{\sqrt{\omega(k)}} \end{pmatrix}; \tag{4}$$

$\varkappa$ and σ are constants depending on $f = \binom{f_0}{f_1}$. From the condition that the functions $f_1(k)$ and

$$\varphi(k) = \omega(k) f_1(k) + \frac{\sigma}{\sqrt{\omega(k)}}$$

have summable squares, just as above in (2), (3), we obtain that $\sigma = 0$. Further, from the self-adjointness of the operator H_r we obtain that $\varkappa = \varkappa(f) = \varkappa_0 f_0$. Thus, the extensions of the operator H_0 differ from the free operator H_0 only by the value of the mass of the V-particle—this is the zero-charge paradox.

We now introduce an indefinite scalar product:

$$(f,f)_R = |f_0|^2 + (f_1,f_1)_R,$$

$$(f_1,f_1)_R = \int_R |f_1(k)|^2\,d^3k = \lim_{M\to\infty} \left(\int_{|k|<M} |f_1(k)|^2\,d^3k - B\ln M \right), \tag{5}$$

$$B = \lim_{M\to\infty} \frac{1}{\ln M} \int_{|k|<M} |f(k)|^2\,d^3k.$$

It is easy to see that the inequalities $|(f_1,f_1)_R| < \infty$ and $|(\varphi,\varphi)_R| < \infty$, where

$$\varphi = \omega f_1 + \frac{\sigma}{\sqrt{\omega}},$$

are not contradictory. Therefore, in the space with scalar product $(f,f)_R$, the operator $\widetilde{H}_0$ has nontrivial extensions. Using the usual rules for constructing extensions, it is not difficult to obtain that all extensions of $\widetilde{H}_0$ are given by the formulas:

$$Hf = \begin{pmatrix} mf_0 + \alpha \\ \omega f_1 + \dfrac{\sigma}{\sqrt{\omega}} \end{pmatrix}, \tag{6}$$

$$\alpha = \mu_{11} f_0 + \mu_{12} \left(\frac{\varphi}{\omega}, \frac{1}{\sqrt{\omega}} \right)_R,$$

$$\varphi = \omega f_1 + \frac{\sigma}{\sqrt{\omega}}; \tag{7}$$

$$\sigma = \mu_{21} f_0 + \mu_{22} \left(\frac{\varphi}{\omega}, \frac{1}{\sqrt{\omega}} \right)_R,$$

μ_{ik} are arbitrary constants.

As can be seen, there are too many extensions. In order to select the needed ones, let us note that by means of formulas (6), (7) the operator $H(m, g, \chi)$ can be specified. In this case

$$\alpha = g \left(f_1, \frac{\chi}{\sqrt{\omega}} \right), \qquad \sigma = g f_0.$$

We rewrite the expression for α with the aid of (7):

$$\alpha = g \left(f_1, \frac{\chi}{\sqrt{\omega}} \right) = g \left(f_1, \frac{\chi}{\sqrt{\omega}} \right)_R = g \left(\frac{\varphi}{\omega} - \frac{\sigma}{\omega\sqrt{\omega}}, \frac{\chi}{\sqrt{\omega}} \right)_R$$

$$= g \left(\frac{\varphi}{\omega}, \frac{\chi}{\sqrt{\omega}} \right)_R - g^2 f_0 \left(\frac{1}{\omega\sqrt{\omega}}, \frac{\chi}{\sqrt{\omega}} \right)_R.$$

Consider the operator $H(m(\chi), g, \chi)$, where

$$m(\chi) = m + g^2 \left(\frac{1}{\omega\sqrt{\omega}}, \frac{\chi}{\sqrt{\omega}} \right)_R.$$

From (1), (6), (7) it is seen that, as $\chi \to 1$, the operator $H(m(\chi), g, \chi)$ has a limit, which is the extension determined by the parameters $\mu_{11} = \mu_{22} = 0$, $\mu_{12} = \mu_{21} = g^*$.

It is not difficult to see that no other extensions can be obtained by a limiting transition from $H(m(\chi), g(\chi), \chi)$ for any dependence of m and g on χ. Thus, an extension is singled out which it is natural to regard as the regularized operator H_r. The operator H_r is therefore determined by formula (6) and the conditions

$$\alpha = g\left(\frac{\varphi}{\omega}, \frac{1}{\sqrt{\omega}}\right)_R, \qquad \sigma = g f_0, \qquad f_1 = \frac{\varphi}{\omega} - \frac{g f_0}{\sqrt{\omega}}, \qquad |(\varphi, \varphi)_R| < \infty.$$

The last equality describes the domain of definition of H_r. Calculations with the operator H_r contain no infinities. Comparison with perturbation theory—calculations shows that the eigenfunctions of the continuous spectrum of H_r are ordinary renormalized eigenfunctions.

Let us note that the operator H_r has two discrete eigenvalues. One of the eigenfunctions of the discrete spectrum has positive norm squared, the other negative. The presence of an eigenfunction with negative norm squared is a consequence of the indefiniteness of the metric. The S-matrix in the sector $\binom{V}{N+\theta}$ can easily be calculated with the aid of the operator H_r and is unitary in the ordinary sense, and not in the sense of the indefinite metric. This property of the S-matrix follows from the fact that it commutes with H_0.

In the sector $\binom{V+\theta}{N+2\theta}$ one can likewise construct the operator H_r by means of the same method as in the sector $\binom{V}{N+\theta}$. In this case the S-matrix turns out to be nonunitary. The nonunitarity of the S-matrix in this sector is a direct consequence of the presence in the sector $\binom{V}{N+\theta}$ of an eigenfunction with negative norm squared.

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* Obviously, the limiting transition $\left(\frac{1}{\omega\sqrt{\omega}}, \frac{\chi}{\sqrt{\omega}}\right)_R \to \left(\frac{1}{\omega\sqrt{\omega}}, \frac{1}{\sqrt{\omega}}\right)_R$ is purely formal.

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