On Domains Star-Shaped with Respect to a Ball

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1962

SovietRxiv

Abstract

Full Text

Mathematics

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(Presented by Academician S. L. Sobolev on February 5, 1962)

The well-known embedding theorems of S. L. Sobolev were proved by him for domains of two types: first, for domains satisfying the "cone condition," i.e., such domains $\Omega \subset R_n$ that at each point of them one can attach, with its vertex lying in Ω, a cone with fixed height and aperture angle; second, for domains star-shaped with respect to some ball contained in them. Subsequently, in works by a number of authors ([1–4]) and others, embedding theorems were proved both for domains of the first type and only for domains of the second type. However, the conjecture had long been expressed that every bounded domain satisfying the "cone condition" can be represented as the sum of a finite number of domains star-shaped with respect to a ball.

The following assertion proves the validity of this conjecture.

Lemma. *If a bounded domain Ω is the sum of an infinite set of domains G_α*

$$\Omega = \bigcup_\alpha G_\alpha,$$

star-shaped with respect to balls of fixed radius $R > 0$ contained in them, then for every $r < R$ there exists a finite number of domains Ω_k $(k = 1, 2, \dots, N)$, star-shaped with respect to balls of radius r contained in them, whose sum coincides with the domain Ω:

$$\Omega = \bigcup_{k=1}^{N} \Omega_k.$$

Proof. Let G_1 be one of the domains of the infinite family of domains G_α. Form the domain

$$\Omega_1 = \bigcup_\beta G_\beta,$$

where the union extends over all domains G_β whose centers of the balls of star-shapedness are at a distance $\rho \leq R - r$ from the center of the ball of star-shapedness of G_1. Obviously, any of the balls of star-shapedness of these domains G_β contains the ball T_1 of radius r with center coinciding with the center of the ball of star-shapedness of G_1. Since each of the domains G_β is star-shaped with respect to this ball, the whole domain Ω_1 is also star-shaped with respect to the ball T_1.

Take as the domain G_2 any one of the domains G_α not included in Ω_1. Repeating the preceding reasoning, construct a domain Ω_2 star-shaped with respect to a ball T_2 of radius r, whose center is at a distance $d > R - r$ from the center of the ball T_1.

After this, in the same way construct a domain Ω_3, star-shaped with respect to a ball T_3 of radius r with center at a distance $d > R - r$ from the centers of the balls T_1 and T_2.

Continue this process further. It is clear that the indicated process will break off after a finite number N of steps, since the centers of the balls T_k are contained in the bounded domain Ω, while the distance between any two of them is greater than the positive number $R - r$.

The lemma is proved.

To prove the possibility of representing a domain Ω satisfying the cone condition as a finite sum of domains star-shaped with respect to ⋯

of the ball of regions, it is enough merely to note that this region is an infinite sum of congruent circular cones and that each of these cones is a region star-shaped with respect to a ball of fixed radius.

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Received
25 XII 1961

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Note: Figure translations are in progress. See original paper for figures.