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Abstract

Full Text

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THE IMPOSSIBILITY OF A STRICTLY TWO-FOLD CONTINUOUS DECOMPOSITION OF A HOMOLOGICAL CUBE

(Presented by Academician P. S. Aleksandrov, 3 I 1962)

1. Theorem. *A two-fold continuous decomposition of a homological cube Q^n contains an element consisting of a single point.*

The case $n = 1$ was analyzed in ⁽¹⁾, the case $n = 2$ in ⁽²⁾, and the Euclidean case $n = 3$ independently in ⁽³⁻⁵⁾. The arguments are applicable to the case of Euclidean space E^n , which, however, is not considered. We note that the example in ⁽⁵⁾ of a strictly two-fold continuous mapping of E^n does not give rise to a continuous decomposition, since the mapping turns out not to be closed.

2. Points belonging to one element of the decomposition are called **conjugate**, and the point conjugate to a point x is denoted by \bar{x} . Continuity of the decomposition means that if a point y is sufficiently close to a point x , then \bar{y} lies in a prescribed neighborhood of the pair $x \cup \bar{x}$.

The closure, open kernel, and boundary of a set M relative to the space X are denoted respectively by: $[M]_X$, $\text{Int}_X M$, $\text{Fr}_X M$.

3,1. The problem was first posed for Euclidean cubes, and for $n \leq 3$ there is no need to appeal to homological manifolds. For $n > 3$ such a need arises because the proof rests on the Smith theorems formulated below. All homology is considered modulo 2. The notion of a generalized homological manifold is taken in Smith's form ⁽⁶⁾; ⁽⁷⁾, p. 268. In ^(8,9) it is shown that such manifolds coincide with Wilder's locally orientable manifolds ⁽¹⁰⁾, p. 281. In what follows they are called ng -manifolds. If an ng -manifold has the homological type of an n -sphere, it is called here an ng -sphere.

3,2. The notion of a generalized homological cube, henceforth simply an ng -cube, is defined in ⁽¹⁰⁾, p. 287. The following assertion holds: if a kg -sphere is placed in a $(k + 1)g$ -sphere, it decomposes it into two regions whose closures are $(k + 1)g$ -cubes, and conversely, the sum of two kg -cubes glued together topologically along their boundaries is a kg -sphere ⁽¹⁰⁾, p. 312.

The boundary of Q^n is denoted by \dot{Q}^n , and $Q^n \setminus \dot{Q}^n$ by $\text{Int } Q$.

3,3. Newman-Smith Theorem ^(11,12); ⁽⁷⁾, p. 272). *The set of fixed points of a continuous involution defined on an ng -manifold is nowhere dense or coincides with the whole manifold.*

3.4. Smith Theorem ⁽⁶⁾; ⁽⁷⁾, p. 272). *The set of fixed points of a continuous involution defined on an n -sphere is a k -sphere, where $k < n$.*

We note that for a Euclidean sphere the set of fixed points need not be a Euclidean sphere (see, for example, ⁽¹⁴⁾).

4. Preliminary construction. Arguing by contradiction, suppose that on Q^n a continuous decomposition φ into pairs of points is given. The closed sets

$$M_n = \{x; d(x, \bar{x}) \geq \frac{1}{n}\}$$

(d is the distance function) then together give Q^n . The open set $H_0 = \bigcup \text{Int } M_n$, by Baire's property, is everywhere dense. $P_1 = Q^n \setminus H_0$ is closed and nowhere dense. Suppose that for all $\alpha < \beta < \omega_1$ closed sets P_α have already been constructed, and each

each P_α is contained and nowhere dense in the preceding ones. If β is a limit ordinal, set $P_\beta = \bigcap P_\alpha$, $\alpha < \beta$; if $\beta = (\beta-1) + 1$, then $P_\beta = P_{\beta-1} \setminus H_{\beta-1}$, where

$$H_{\beta-1} = \bigcup \text{Int}_{\beta-1}(M_n \cap P_{\beta-1}).$$

P_β is closed and nowhere dense in all P_α , $\alpha < \beta$. For some $\gamma < \omega_1$ the set P_γ is empty, and Q^n is the sum of pairwise disjoint sets H_α , $0 \leq \alpha < \gamma$.

5. Lemmas

Lemma 1. *If $x \in P_1$, then every neighborhood of x contains conjugate pairs; if, however, $x \in H_0$, then there is a neighborhood of x containing no conjugate pairs.*

Lemma 2. *If $x_0 \in H_0$, then the mapping $x \rightarrow \bar{x}$ maps some neighborhood of x_0 topologically onto some neighborhood of \bar{x}_0 .*

Both lemmas follow easily from the continuity of the decomposition φ and the definition of the set H_0 .

Lemma 3. *If x and \bar{x} lie in H_0 , then they are either both boundary points or both interior points.*

Indeed, at boundary points and at interior points Q^n has different local homological character, whereas, according to Lemma 2, x and \bar{x} have homeomorphic neighborhoods.

Lemma 4. *If $x_0 \in P_1$, then every neighborhood of x_0 contains conjugate pairs both of whose points belong to $H_0 \cap \text{Int } Q^n$.*

Let $x_0 \in H_1$; since $[H_1] = P_1$, it is enough to consider this case. Let

$$0 < \varepsilon < \frac{1}{2}d(x_0, P_2)$$

and choose δ , $0 < \delta < \varepsilon$, so that if

$$x \in O_\delta(x_0) \cap H_1,$$

then

$$\bar{x} \in O_\varepsilon(\bar{x}_0).$$

In $O_\delta(x_0)$, according to Lemma 1, there must be a conjugate pair. By the choice of ε , only points of H_0 and H_1 can enter it. But, by the choice of δ , points of H_1 are excluded. Since H_0 is open and, by Lemma 3, such a pair may be taken in $\text{Int } Q^n$.

6. Main lemma

The mapping $e : Q^n \rightarrow Q^n$, defined by

$$e(x) = \bar{x}, \quad \text{if } x \in H_0; \quad e(x) = x, \quad \text{if } x \in P_1,$$

is a continuous involution of Q^n .

First one proves the continuity of e at interior points, then at boundary points, and finally that if $x \in H_0$, then also $\bar{x} \in H_0$. At points of H_0 , e is continuous by Lemma 2. Suppose that for all $\alpha < \beta$, e is continuous at the points of

$$H_\alpha \cap \text{Int } Q^n$$

and let

$$x_0 \in H_\beta \cap \text{Int } Q^n.$$

We show that e is also continuous at the point x_0 . Let

$$0 < \varepsilon < \frac{1}{3}d(x_0, \bar{x}_0 \cup P_{\beta+1} \cup \dot{Q}^n).$$

There is δ_1 such that

$$0 < \delta_1 < \varepsilon$$

and

$$\text{if } x \in O_{\delta_1}(x_0) \cap H_\beta, \quad \text{then } \bar{x} \in O_\varepsilon(\bar{x}_0). \quad (1)$$

By the continuity of φ , there is δ , $0 < \delta < \delta_1$, such that

$$\text{if } x \in O_\delta(x_0), \quad \text{then } \bar{x} \in O_{\delta_1}(x_0) \cup O_\varepsilon(\bar{x}_0). \quad (2)$$

By virtue of (2), $H_0 \cap O_\delta(x_0)$ splits into the set Φ_1 of points conjugate to which lie in $O_{\delta_1}(x_0)$, and the set Φ_2 of points conjugate to which lie in $O_\varepsilon(\bar{x}_0)$. Both are open. Φ_1 , in turn, splits into the set Φ'_1 of points conjugate to which lie in $O_{\delta_1}(x_0) \cap H_0$, and the set Φ''_1 of points conjugate to which lie in $O_{\delta_1}(x_0) \cap P_1$. Φ'_1 is open by Lemma 2. Φ''_1 is also open. Otherwise, arbitrarily close to $x \in \Phi''_1$ there would lie points of Φ'_1 , and e would have a discontinuity at \bar{x} . But e is

continuous at \bar{x} by the induction hypothesis, since, by (1), \bar{x} cannot belong to H_β .

Since the sets $\Phi'_1, \Phi''_1, \Phi_2$ are open,

$$\text{Fr}_{O_\delta} \Phi'_1 \cup \text{Fr}_{O_\delta} \Phi''_1 \cup \text{Fr}_{O_\delta} \Phi_2 \subset P_1. \quad (3)$$

Define on $O_\delta(x_0)$ a mapping g by the equalities:

$$g(x) = \dot{x}, \quad \text{if } x \in \Phi''_1 \cup \Phi_2, \quad (4)$$

$$g(x) = e(x), \quad \text{if } x \in \Phi'_1 \cup \Phi_2. \quad (5)$$

It is continuous; in view of (3), it is enough to verify this only at the points $\text{Fr}_{O_\delta} \Phi'_1$. If $x \in \text{Fr}_{O_\delta} \Phi'_1 \cap H_\alpha$, where $\alpha < \beta$, then g is continuous at x by (5) and the induction hypothesis. If, however, $x \in \text{Fr}_{O_\delta} \Phi'_1 \cap H_\beta$, then g is continuous at x by the definition of the set Φ'_1 and the continuity of the partition φ . By (4), g has period 2 on $O_\delta(x_0) \cap g(O_\delta(x_0))$. A connected neighborhood O such that $g(O) = O \subset O_\delta \cap g(O_\delta)$ satisfies the condition of the Newman–Smith theorem, and, since $\Phi_1 \cap O \neq \Lambda$, in this neighborhood the set of fixed points of g is nowhere dense. Consequently, $O \cap (\Phi_1 \cup \Phi_2) = \Lambda$. Since e coincides with g in O , it is continuous on O , hence at x_0 , and hence everywhere inside Q^n .

From the emptiness of $\Phi''_1 \cup \Phi_2$ in O there follows the following observation, used below.

Remark. If a point $x \in H_0$ is sufficiently close to $P_1 \cap \text{Int } Q^n$, then \bar{x} also belongs to H_0 .

Now let $x_0 \in P_1 \cap \dot{Q}^n$, $0 < \varepsilon < \frac{1}{3}d(x_0, \bar{x}_0)$, and choose δ , $0 < \delta < \varepsilon$, so that if $x \in O_\delta(x_0)$, then $\bar{x} \in O_\varepsilon(x_0 \cup \bar{x}_0)$. $H_0 \cap O_\delta$ decomposes, therefore, into two open sets: Φ_1 of points whose conjugates lie in $O_\varepsilon(x_0)$, and Φ_2 of points whose conjugates lie in $O_\varepsilon(\bar{x}_0)$. Φ_1 is nonempty by Lemma 4. If e has a discontinuity at x_0 , then Φ_2 is also nonempty. In $O_\delta(x_0)$ the sets Φ_1 and Φ_2 are separated by points of P_1 . But at a point $y \in \text{Fr}_{O_\delta} \Phi_2 \cap P_1 \cap \text{Int } Q^n$ the mapping e would be discontinuous, which, as proved, is impossible at an interior point. Thus e is continuous everywhere in Q^n .

It remains to show that if $x \in H_0$, then also $\bar{x} \in H_0$. Assuming the contrary, consider the set Φ of points x such that $x \in H_0$, and $\bar{x} \notin H_0$. Let $x_0 \in \Phi$. Then an entire neighborhood of x_0 lies in Φ ; otherwise e would have a discontinuity at \bar{x}_0 . Hence Φ is open. Using the observation made above, we find that $\text{Fr } \Phi \cap P_1 \cap \text{Int } Q^n$ is empty. Consequently, $\Phi \supset \text{Int } Q^n$. But this contradicts Lemma 4. Therefore Φ is empty and e is a continuous involution of Q^n .

7. Proof of the theorem. Let $C = \dot{Q}^n \setminus p$ be the cone over the boundary of Q^n . Lemma 3 permits one to extend e , in the following way, to an involution on

$S = Q^n \cup C$. If $x \in \dot{Q}^n \cap P_1$, the whole segment $[px]$ remains fixed; if $x \in \dot{Q}^n \cap H_0$, then the segment $[px]$ is mapped linearly onto the segment $[p\bar{x}]$. In ⁽¹³⁾ it is shown that the cone over an $(n-1)$ g -sphere is an ng -cube, and, according to 3.2, S is an ng -sphere. By 3.4 the set of fixed points K is a kg -sphere, and the following cases are possible: $k = -1, 0, 1, \dots, n-1$. The first case $k = -1$ is impossible, since p is fixed. The second case $k = 0$, a pair of points, means that P_1 consists of a single point, which therefore has no conjugate, contrary to the supposition. The remaining cases $k = 1, \dots, n-1$ are reduced to the preceding one as follows. $C_1 = C \cap K$ is a cone over its boundary in K . Using Theorem 2 from ⁽¹³⁾, we conclude that this boundary is a $(k-1)$ g -sphere, and, consequently, according to 3.2, $P_1 = K \setminus C_1$ is a kg -cube. By the main lemma, φ induces on P_1 a strictly twofold continuous partition. Therefore all the arguments carried out for Q^n are applicable to P_1 . Since $k < n$, after a finite number of steps we arrive at the case $k = 0$, and hence at a contradiction.

8. Remark 1. The theorem is valid, in particular, for the Euclidean cube, and also remains true for $S^n \setminus p$, where S^n is an ng -sphere and $p \in S^n$, in particular for E^n .

Remark 2. The arguments carried out in the proof of the main lemma show that every strictly twofold continuous decomposition φ defined on an ng -sphere S^n is determined by a sequence of embedded kg -spheres

$$S^n \supset S^{k_1} \supset \dots \supset S^{k_m},$$

where $n > k_1 > \dots > k_m \geq 0$, and by involutions $\varphi_0, \varphi_1, \dots, \varphi_m$, defined respectively on the spheres S^{k_i} , with S^{k_i+1} serving as the set of fixed points for φ_{k_i} . The elements of the decomposition φ are the pairs of points corresponding to one another under a certain involution.

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