DEVIATION OF HARMONIC FUNCTIONS FROM THEIR VALUES ON THE BOUNDARY AND BEST APPROXIMATION

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Abstract

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DEVIATION OF HARMONIC FUNCTIONS FROM THEIR VALUES ON THE BOUNDARY AND BEST APPROXIMATION

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In this note we study functions harmonic in a disk or half-plane, possessing prescribed constructive properties on the boundary, and investigate the order of magnitude of the deviation of such functions from their values on the boundary.

For definiteness, consider the disk of unit radius centered at the origin and a function $U(r;x)$ $(0 \le r < 1;\ 0 \le x \le 2\pi)$, harmonic in this disk, whose values on the boundary coincide with the values of a given integrable function $f(x)$. Under the assumption that, for some $1 \le p \le \infty$,

$$\|f(x)\|_{L_p} = \left\{ \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p} < \infty \qquad (1 \le p < \infty),$$

$$\|f(x)\|_{L_p} = \operatorname*{Vrai\,sup}_{0 \le x < 2\pi} |f(x)| < \infty \qquad (p = \infty),$$

estimates are given for the magnitude of the deviation

$$\Delta(f;r)_{L_p} = \|f(x) - U(r;x)\|_{L_p} \qquad (1 \le p \le \infty)$$

in terms of the sequence $\{E_n(f)_{L_p}\}$ of best approximations of the function $f(x)$ by trigonometric polynomials of order $\le n$,

$$E_n(f)_{L_p} = \inf_{\alpha_k, \beta_k} \left\| f(x) - \sum_{k=0}^{n} (\alpha_k \cos kx + \beta_k \sin kx) \right\|_{L_p} \qquad (n = 0, 1, 2, \ldots).$$

Theorem 1. If $f(x) \in L_p$ $(1 \le p \le \infty)$, then

$$\Delta(f;r)_{L_p} \le C(1-r) \sum_{n=0}^{\infty} r^n E_n(f)_{L_p} \qquad (1 \le p \le \infty), \tag{1}$$

where C is an absolute constant.

Inequality (1), in the general case for $p=\infty$ and $p=1$, cannot be improved in order. However, for $1<p<\infty$, using the results of papers ([2],[3],[6]) and the method applied by the author in ([4]), one can give sharper estimates for the order of decrease of the quantity $\Delta(f;r)_{L_p}$ as $r\to 1$.

Theorem 2. If $f(x)\in L_p,\ 1<p<\infty$, then

$$\Delta(f;r)_{L_p}\le C_p(1-r)\left\{\sum_{n=0}^{\infty} r^{n\gamma}(n+1)^{\gamma-1}E_n^{\gamma}(f)_{L_p}\right\}^{1/\gamma}, \tag{2}$$

where $\gamma=p$ for $1<p\le 2$ and $\gamma=2$ for $2\le p<\infty$.

As a consequence of Theorems 1 and 2 we obtain, for example, that if

$$E_n(f)_{L_p}=O\left(\frac{1}{n}\right),$$

then

$$\Delta(f;r)_{L_p}=O\begin{cases}(1-r)\left(\ln\dfrac{1}{1-r}\right)^{1/p}, & (1\le p\le 2),\\ (1-r)\left(\ln\dfrac{1}{1-r}\right)^{1/2}, & (2\le p<\infty),\\ (1-r)\ln\dfrac{1}{1-r}, & (p=\infty),\end{cases}\qquad (r\to 1). \tag{3}$$

Inequality (2) in the general case cannot be improved in order.

Analogous results hold for functions $U(x,y)$ harmonic in a half-plane. For definiteness, the upper half-plane $(y\geqslant 0)$ is considered. Under the assumption that the boundary values $f(x)$ of the function $U(x,y)$ for some $1\leqslant p\leqslant\infty$ satisfy the condition

$$\|f(x)\|_{L_p}=\left\{\int_{-\infty}^{\infty}|f(x)|^p\,dx\right\}^{1/p}\qquad (1\leqslant p<\infty),$$

$$\|f(x)\|_{L_p}=\operatorname*{Vrai\,sup}_{-\infty<x<\infty}|f(x)|<\infty\qquad (p=\infty),$$

for the magnitude of the deviation

$$\Delta(f;y)_{L_p}=\|U(x,y)-f(x)\|_{L_p}$$

the following estimates are valid:

Theorem 3. If $f(x) \in L_p$ $(1 \leqslant p \leqslant \infty)$, then

$$\Delta(f;y)_{L_p} \leqslant Cy \int_0^{1/y} A_\sigma(f)_{L_p}\, d\sigma, \tag{4}$$

where C is an absolute constant, and

$$A_\sigma(f)_{L_p} = \inf_{Q_\sigma} \|f(x) - Q_\sigma(x)\|_{L_p},$$

$Q_\sigma(x)$ is an entire function of degree not exceeding σ, satisfying the condition

$$\int_{-\infty}^{\infty} |Q_\sigma(x)|^p\, dx < \infty.$$

Theorem 4. If $f(x) \in L_p$ $(1 < p < \infty)$, then

$$\Delta(f;y)_{L_p} \leqslant C_p y \left\{ \int_0^{1/y} (\sigma+1)^{\gamma-1} A_\sigma^\gamma(f)_{L_p}\, d\sigma \right\}^{1/\gamma}, \tag{5}$$

where $\gamma = p$ for $1 < p \leqslant 2$ and $\gamma = 2$ for $2 \leqslant p < \infty$.

The proof of Theorems 1 and 3 is carried out with the aid of known integral representations of functions harmonic in the domains under consideration, by applying inverse theorems of the constructive theory of functions (see ([1]), Ch. 6). In proving Theorem 4, some results from the author’ s paper ([5]) are used.

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Note: Figure translations are in progress. See original paper for figures.