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L. S. SOLOV'EV

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Abstract

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L. S. SOLOV' EV

ON THE STABILITY OF A CYLINDRICAL PLASMA JET IN A MAGNETIC FIELD

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1. In the paper [1], the following system of magnetohydrodynamic equations was obtained for stationary helical flows of an ideally conducting incompressible fluid in a magnetic field:

$$s\Delta^*\xi + \frac{s'}{2\beta}(\nabla\xi)^2 + \frac{1}{2\beta}(a^2 s)' + \frac{\beta}{2}(b^2 s)' - \frac{2\alpha}{\beta^2}as + u' = 0; \qquad (1)$$

$$s = \psi_0'^2 - \psi'^2, \qquad p + \frac{v^2}{2} = -u - s\beta b^2; \qquad (2)$$

$$\begin{pmatrix} v_r \\ H_r \end{pmatrix} = \frac{1}{r}\frac{\partial\xi}{\partial\theta}\begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix}, \qquad \alpha r\begin{pmatrix} v_z \\ H_z \end{pmatrix} - \begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = \frac{\partial\xi}{\partial r}\begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix},$$

$$\begin{pmatrix} v_z \\ H_z \end{pmatrix} + \alpha r\begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = a\begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix} + \beta b\begin{pmatrix} \psi' \\ \psi_0' \end{pmatrix}. \qquad (3)$$

Here r, φ, z are cylindrical coordinates; $\theta = \varphi - \alpha z$; $\beta = 1 + \alpha^2 r^2$;

$$\Delta^* = \frac{1}{r}\frac{\partial}{\partial r}\frac{r}{\beta}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}; \qquad \alpha = \frac{2\pi}{L};$$

L is the pitch of the helix; $\mathbf{v}$ is the velocity of the fluid; $\mathbf{H} = \mathbf{B}/\sqrt{4\pi}$; $\mathbf{B}$ is the magnetic field; p is the pressure, and the density ρ is taken equal to unity. The quantities ψ_0, ψ, a, b, and u are arbitrary functions depending only on ξ, and the prime denotes differentiation with respect to ξ. The equation $\xi(r, \theta) = \text{const}$ describes a family of magnetic surfaces possessing helical symmetry.

Steady-state waves considered in a moving coordinate system are a special case of stationary flows. The equation for helical waves in a circular plasma cylinder is singled out from equation (1) by an appropriate choice of the functions s, a, b, and u, which are determined by the equilibrium distribution of the velocity $\mathbf{v}(r)$ and magnetic field $\mathbf{B}(r)$, and are chosen so that equation (1), along with helical solutions, would also have the solution $\xi = \xi(r)$, describing an unperturbed state with cylindrical magnetic surfaces. In deriving the wave equation, one uses the invariance of the functions of ξ with respect to deformations of the magnetic surfaces, i.e., the transition from the unperturbed to the perturbed state. The wave equation obtained in this way is, generally speaking, nonlinear.

If the magnetic field outside the plasma is also described by the corresponding current function $\psi_e(r,\theta) = A_z + \alpha r A_\varphi$, then the boundary conditions of tangency of the lines of force are written in the form $\xi|_\Sigma = \text{const}$, $\psi_e|_\Sigma = \text{const}$. The third boundary condition for ξ and ψ_e, following from the balance of pressures at the free surface of the plasma, $2p + H^2 = H_e^2$, is always nonlinear.

Here we shall give only the linearized equations and boundary conditions for waves in a plasma jet. If ξ and ψ_e are represented in the form

$$\xi = \frac{r^2}{2} + \tilde{\xi}, \quad \psi_e = H_{ze}\frac{\alpha r^2}{2} - H_{\varphi e}\ln r + \tilde{\psi}_e, \quad \text{where } \tilde{\xi}_e = f_m(r)e^{im\theta}, \quad \tilde{\psi}_e =$$

$= f_{me}(r)e^{im\theta}$, then the equation for f_{me} and the equation linearized (with respect to $\tilde{\xi}$) for f_m take the form

$$\left(\frac{r}{\beta}f'_{me}\right)' - \frac{m^2}{r}f_{me} = 0; \tag{4}$$

$$\left(\frac{r}{\beta}f'_m\right)' + \left\{-\frac{m^2 s}{r} + \frac{4\alpha^2 r a_1^2}{\beta s} + \left(\frac{2a_1}{\beta} + v_\varphi^2 - h_\varphi^2\right)'\right\} f_m = 0. \tag{5}$$

The boundary condition that must be satisfied on the unperturbed boundary of the plasma cylinder $r = R$ is obtained by linearization with respect to $\tilde{\xi}$ and ψ_e:

$$s\frac{Rf'_m}{f_m} + J_e^2\frac{Rf'_{me}}{f_{me}} + \beta\left(\frac{2a_1}{\beta} + v_\varphi^2 - h_\varphi^2 + h_{\varphi e}^2\right) = 0. \tag{6}$$

Here $s = J_0^2 - J^2$; $a_1 = J_0 v_\varphi - J h_\varphi$; $J_0 = \alpha(v_z + v_\Phi) - v_\varphi$; $J = \alpha H_z - h_\varphi$; $r v_\varphi = v_\phi$; $r h_\varphi = H_\phi$; $\mathbf{v}(r)$ and $\mathbf{H}(r)$ are prescribed (equilibrium) functions of the radius; $v_\Phi = \omega/k$ is the phase velocity of the wave. The parameter α is related to the wave number $k = 2\pi/\lambda$ by the relation $\alpha m = k$. Analogous equations describing the propagation of linear waves in a plasma cylinder at rest ($v_z = v_\varphi = 0$) were obtained in papers ([2], [3]).

2. In the case of a plasma cylinder rotating as a whole ($v_\varphi = \text{const}$) with a uniform longitudinal current ($h_\varphi = \text{const}$) and a uniform longitudinal magnetic field ($H_z = \text{const}$), equation (5) is exact (independently of the perturbation amplitude), and the problem of stationary helical waves

in such a plasma cylinder bounded by an ideally conducting wall can be solved exactly. In this case equation (5) has the solutions $f_m(r) \sim \varepsilon J_m(\chi r) - \alpha r\chi J'_m(\chi r)$, where $J_m(x)$ is the Bessel function, $\chi^2 = \varepsilon^2 - \alpha^2 m^2$, $\varepsilon = 2\alpha a_1/s$. Denoting by $i_n = (\chi R)_n$ the roots of $f_m(R)$*, from the boundary condition $f_m(R) = 0$ we find

$$v_\Phi = v_\varphi \left(\frac{1}{\alpha} + \frac{R}{i_n}\right) \pm \frac{1}{\alpha}\sqrt{J^2 - \frac{2\alpha R h_\varphi}{i_n} J + \frac{\alpha^2 R^2 v_\varphi^2}{i_n^2}}, \tag{7}$$

where $j_n^2 = i_n^2 + \alpha^2 m^2 R^2$.

If the expression under the radical is negative, the phase velocity contains both a real and an imaginary part, i.e., an oscillatory-type instability occurs ([4]). The stability condition is $v_\varphi^2 \geq h_\varphi^2$.

In the equilibrium plasma configuration under consideration the pressure is distributed according to a parabolic law. From the equilibrium condition

$$\left(p + \frac{H^2}{2}\right)' = r\left(v_\varphi^2 - h_\varphi^2\right) \tag{8}$$

there follows the possibility of a pressure decreasing with radius if $v_\varphi^2 < 2h_\varphi^2$. Thus, for

$$B_\varphi^2 \leq 4\pi\rho v_\varphi^2 < 2B_\varphi^2 \tag{9}$$

a plasma cylinder with pressure (and therefore also temperature) going to zero at $r = R$ is stable with respect to arbitrary helical perturbations.

3. Let us now consider the case of an arbitrary distribution of the magnetic field $\mathbf{B}(r)$ and suppose that $\mathbf{v} \parallel \mathbf{B}$. Eliminating the term with the first derivative by means of the substitution $F = \sqrt{rs/\beta}\, f_m$ and expanding the equa-

* For long waves ($kR \ll 1$), i_n are close to the roots of $J_m(x)$.

expanding (5) in the neighborhood of the point $r = r_s$, where $J = 0$, we obtain

$$\frac{d^2F}{d\xi^2} + \left\{-E - \frac{N}{(1+\xi^2)^2} + \frac{M}{1+\xi^2}\right\} F = 0. \tag{10}$$

Here

$$\xi = \frac{r - r_s}{\gamma} + i\frac{J'_0}{J'}, \quad \gamma^2 = -\left(\frac{\alpha v_\phi J'}{J'^2 - J_0'^2}\right)^2; \quad E = \frac{m^2}{r_s^2}\gamma^2; \quad N = 1 + 4\alpha^2 h_\phi^2 / J'^2;$$

$$M = -\frac{2\alpha^2(p + v^2/2)'}{r_s({J'}^2 - J_0'^2)}.$$

In deriving (10) it was assumed that v_ϕ is imaginary and sufficiently small in absolute value, $m \gg 1$, and the equilibrium equation (8) was used. If equation (10) has a localized solution for $E > 0$, then the plasma cylinder is unstable ($v_\phi^2 < 0$) independently of the boundary conditions. For real $\xi = x$, the condition for the existence of eigenvalues $E > 0$ is $M > 1/4$ ([5]). The local solution on the real axis x can be analytically continued into the complex ξ-plane. Thus, for $M > 1/4$, local solutions also exist for $\xi = x + iJ_0'/J'$, if $|J_0'| \ll |J'|$, i.e., if the parallel displacement of the axis is smaller than the distance to the singular points $\pm i$. The latter condition reduces to the requirement $4\pi\rho v^2 < B^2$. In expanded form the stability criterion $M < 1/4$ has the form

$$\frac{8\pi}{B_z^2}\left(p + \rho\frac{v^2}{2}\right)' + \frac{r}{4}\left(\frac{\mu'}{\mu}\right)^2\left(1 - \frac{4\pi\rho v^2}{B^2}\right) > 0, \tag{11}$$

where $\mu = B_\phi/rB_z$. For $v = 0$, (11) becomes Suydam' s criterion ([6]), and for $B_\phi \sim r,\ B_z = \text{const}$, it becomes the condition (9) obtained above.

4. For axisymmetric oscillations ($m = 0$) of a rotating plasma cylinder without a longitudinal field ($B_z = 0$), according to (5) we have the equation

$$\left(\frac{1}{r}f_0'\right)' + \left\{-\frac{k^2}{r} + \frac{k^2}{\omega^2}\left[\frac{(r^4 v_\phi^2)'}{r^4} - (h_\phi^2)'\right]\right\} f_0 = 0. \tag{12}$$

From the general Sturm–Liouville theory ([7]) it follows that the necessary and sufficient condition for the existence of $\omega^2 > 0$ for the boundary-value problem $f_0(0) = f_0(R) = 0$ is the positivity of the expression in square brackets. Hence we obtain the stability condition

$$(\rho r^2 v_\phi^2)' - \frac{r^4}{4\pi}\left(\frac{B_\phi^2}{r^2}\right)' > 0, \tag{13}$$

which for $B_\phi = 0$ becomes Rayleigh' s criterion ([9]), and for $B_\phi \sim r,\ v_\phi \sim r$ becomes the condition $v_\phi^2 > 0$, which also follows from the solution of the exact problem in Sec. 2, if $B_z = 0$. Criterion (13), like (11), is local in character and can be established on the basis of an analysis of the balance of forces ([4]).

5. Condition (11) is derived from equation (5) without taking into account the boundary condition (6). On the other hand, from (6), in the case of continuity of the fields B_z and B_ϕ at the surface of the plasma cylinder bordering the external magnetic field, one can obtain a sufficient condition

for instability without specifying the solutions of (5). If at the boundary of the cylinder $r = R$ the quantities J and v_ϕ are small, then, neglecting their squares, we obtain from (6)

$$J_0 = -\frac{v_\phi f_m}{R f'_m} \pm \sqrt{\frac{2h_\phi J f_m}{R f'_m}}. \tag{14}$$

It follows from this that the boundary of the stability region for the mode m is $J = 0$, or

$$B_z = \frac{m B_\phi}{kR}. \tag{15}$$

For a plasma cylinder with a uniform longitudinal field and a uniform current, the condition for the occurrence of instability (15) was obtained in work ([8]); however, in this particular case the instability also follows from Suydam' s criterion.

The surface instability considered arises when the helical perturbation of the plasma surface is directed along the magnetic-field lines. It is associated only with the existence of a free boundary of the plasma cylinder and takes place independently of the distribution of the internal magnetic field and, consequently, of the fulfillment of Suydam' s criterion.

In conclusion, let us note that consideration of nonlinear helical waves shows that, on approaching the boundary of the stability region (15), there is a tendency for the wave crests on the surface of the plasma cylinder to sharpen.

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Note: Figure translations are in progress. See original paper for figures.