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Abstract

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MATHEMATICS

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ON THE CORRESPONDENCE OF BOUNDARIES UNDER Q-QUASICONFORMAL MAPPINGS OF A BALL

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Here we consider homeomorphic Q-quasiconformal (^{1}) mappings of a ball in n-dimensional ($n \geq 2$) Euclidean space E^n, and prove a theorem analogous to Koebe's theorem (^{2}) on the correspondence of attainable boundary points under conformal mappings of the disk onto the plane.

Lemma 1. *Let $\{c\}$ be the family of all curves joining in the ball $R^n \subset E^n$ nonintersecting sets l_1, l_2; let $\{P_h\}$ $(0 \leq h \leq H)$ be a one-parameter family of parallel $(n-1)$-dimensional sections of the ball R^n, intersecting both sets l_1, l_2 and filling a spherical layer V^n of height H. Then for the modulus (^{1}) $M\{c\}$ of the family $\{c\}$ the following lower estimate holds*

$$M\{c\} \geq k(n)\frac{H}{d},$$

where $k(n)$ is a constant depending only on n; $d = \sup\limits_{0 \leq h \leq H} \rho(l_1 \cap P_h, l_2 \cap P_h)^*$.

For simplicity we shall give the proof for $n = 3$; the general case is essentially no different.

Let $\eta(p) \geq 0$ be an admissible metric of the family $\{c\}$ (see (^{1})). Then

$$\iiint_{R^3} \eta^3 \, d\omega \geq \iiint_{V^3} \eta^3 \, d\omega = \int_0^H \left[\iint_{P_h} \eta^3 \, d\sigma \right] dh. \tag{1}$$

Let us estimate the inner integral on the right-hand side of (1). Take two points $p_i = p_i(h) \in l_i \cap P_h$ $(i = 1, 2)$, and in the section P_h construct the isosceles right triangle Δ_h with hypotenuse $\overline{p_1 p_2}$. The straight line $l \subset P_h$, orthogonal to the segment $\overline{p_1p_2}$ and passing through its midpoint, divides Δ_h into two triangles Δ_h^1, Δ_h^2 symmetric with respect to l, and

$$\iint_{R_h} \eta^3\, d\sigma \geq \iint_{\Delta_h} \eta^3\, d\sigma = \iint_{\Delta_h^1} \eta^3\, d\sigma + \iint_{\Delta_h^2} \eta^3\, d\sigma. \tag{2}$$

In the section P_h introduce two systems of polar coordinates (p_1), (p_2) with poles at the points p_1 and p_2, respectively. In the first of these systems we shall measure the polar angle from the direction $\overrightarrow{p_1p_2}$ counterclockwise, and in the second–from the direction $\overrightarrow{p_2p_1}$ clockwise. Under these conditions the points of the line l will have the same coordinates in both systems, and the line l itself is written in the form $r = \rho/2\cos\varphi$, $|\varphi| < \pi/2$, where $\rho = \rho(p_1, p_2)$.

Let $\eta_i(r, \varphi)$ be the value of the function $\eta(p)$ at the point determined by the coordinates (r, φ) in the corresponding polar system (p_i) $(i = 1, 2)$; then

$$\iint_{\Delta_h^i} \eta^3\, d\sigma = \int_0^{\pi/4} d\varphi \int_0^{\rho/2\cos\varphi} \eta_i^3(r, \varphi)\, r\, dr \qquad (i = 1, 2).$$

$*$ By $\rho(M_1, M_2)$ is denoted the Euclidean distance between the sets M_1 and M_2.

Using Hölder' s inequality, we find

$$\iint_{\Delta_h^i} \eta^3\, d\sigma \geq \int_0^{\pi/4} \frac{2\cos\varphi\, d\varphi}{4\rho} \left[\int_0^{\rho/2\cos\varphi} \eta_i(r, \varphi)\, dr\right]^3 \qquad (i = 1, 2),$$

$$\iint_{\Delta_h = \Delta_h^1 + \Delta_h^2} \eta^3\, d\sigma \geq \frac{\sqrt{2}}{4\rho(p_1, p_2)} \int_0^{\pi/4} \left[\left(\int_0^{\rho/2\cos\varphi} \eta_1(r, \varphi)\, dr\right)^3 + \left(\int_0^{\rho/2\cos\varphi} \eta_2(r, \varphi)\, dr\right)^3\right] d\varphi. \tag{3}$$

Since $\eta(p) \geq 0$ is an admissible metric of the family $\{c\}$, we have

$$\int_0^{\rho/2\cos\varphi} \eta_1(r, \varphi)\, dr + \int_0^{\rho/2\cos\varphi} \eta_2(r, \varphi)\, dr \geq 1.$$

Taking also into account that $a^n + b^n \geq 2^{(1-n)}(a + b)^n$ for $a \geq 0$, $b \geq 0$, from (3) we obtain

$$\iint_{\Delta_h} \eta^3\, d\sigma \geq \pi\sqrt{2}/64\, \rho(p_1(h), p_2(h)).$$

Since $p_1(h), p_2(h)$ are arbitrary points of intersection $l_1 \cap P_h$, $l_2 \cap P_h$, we conclude that

$$\iint_{\Delta_h} \eta^3 \, d\sigma \geq \frac{\pi\sqrt{2}}{64} \frac{1}{\rho(l_1 \cap P_h,\ l_2 \cap P_h)},$$

and, returning to (2), and then to (1), we finally find

$$\iiint_{R^3} \eta^3 \, d\omega \geq \frac{\pi\sqrt{2}}{64} \frac{H}{d}.$$

Since η is an arbitrary admissible metric, Lemma 1 is proved.

On the basis of Lemma 1 the following is proved.

Lemma 2. *If $\{c\}$ is the family of all possible curves joining in the ball R^n two connected sets that have a common limit point, then $M\{c\} = \infty$.*

We now introduce two definitions. Let d be the lower bound of the lengths of all possible curves joining in a domain D the sets $A \subset D$ and $B \subset D$; let D', A', B' be the images of D, A, and B, respectively, under inversion with respect to some nondegenerate sphere of finite radius; let d' be the lower bound of the lengths of all possible curves joining in D' the sets $A' \subset D'$ and $B' \subset D'$.

Definition 1. The quantity $\rho_D(A, B)$, equal to d when $d' > 0$ and equal to 0 when $d' = 0$, will be called the **distance** between the sets A and B relative to the domain D.

Definition 2. Two curves $\lambda_1 \subset D$ and $\lambda_2 \subset D$, going to one and the same point* p of the boundary of the domain D, will be regarded as **D-equivalent** if, in every neighborhood U_p of the point p,

$$\rho_D(\lambda_1 \cap U_p, \lambda_2 \cap U_p) = 0.$$

On the basis of Lemma 2 one easily obtains:

Lemma 3. *Let $p^* = T(p)$ be a homeomorphic Q-quasiconformal mapping of the ball R^n onto a domain D^*, under which connected sets $A \subset R^n$ and $B \subset R^n$ pass respectively into the sets $A^* \subset D^*$ and $B^* \subset D^*$. If*

$$\rho_{R^n}(A, B) = 0,$$

then also $\rho_{D^*}(A^*, B^*) = 0$.

From this lemma it follows immediately that:

Lemma 3′. *Let $p^* = T(p)$ be a homeomorphic Q-quasiconformal mapping of the ball R^n onto a domain D^*, and suppose that under this mapping R^n-equi-*

ivalent curves $\lambda_1 \subset R^n$, $\lambda_2 \subset R^n$ pass respectively into the curves $\lambda_1^* \subset D^*$, $\lambda_2^* \subset D^*$. If the mapping $p^* = T(p)$ has a limit as the point p approaches the boundary of R^n along each of the curves λ_1, λ_2, then the curves λ_1^*, λ_2^* are D^*-equivalent.

Lemma 4. If $p^* = T(p)$ is a homeomorphic Q-quasiconformal mapping of the ball R^n, then in R^n one cannot select a sequence of arcs γ_m whose ends converge to two distinct points of the boundary sphere Γ and whose images $\gamma_m^* = T(\gamma_m)$ shrink to a point.

Proof. Suppose that, contrary to the assertion of the lemma, it has been possible to select in R^n a sequence of arcs γ_m whose ends converge to two distinct points $p_1 \in \Gamma$, $p_2 \in \Gamma$, and whose images $\gamma_m^* = T(\gamma_m)$ shrink to the point p_0^*. Let l be a segment of length $\rho(p_1, p_2)/2$, parallel to the segment $\overline{p_1 p_2}$ and symmetric with respect to the center of the ball R^n. Consider the sequence of families $\{c_m\}$ of all curves joining in R^n the segment l with the corresponding arc γ_m. Without loss of generality, the ends of the arcs γ_m may be assumed so close to the points p_1, p_2, respectively, that each of the two $(n-1)$-dimensional planes drawn orthogonally to the segment l through its ends intersects all the arcs γ_m. Then, by Lemma 1, for any family $\{c_m\}$ we have:

$$M\{c_m\} \geqslant k(n)\frac{\rho(p_1, p_2)}{2r} = \varepsilon^{2(n-1)} > 0,$$

where r is the radius of the ball R^n.

Since the mapping $p^* = T(p)$ is Q-quasiconformal, the following estimate (1) must hold for the modulus $M\{c_m^*\}$ of the image of the family $\{c_m\}$:

$$M\{c_m^*\} \geqslant Q^{-(n-1)} M\{c_m\} \geqslant \frac{\varepsilon^{2(n-1)}}{Q^{(n-1)}} \qquad (m = 1, 2, \ldots). \tag{4}$$

Now let p_0^* be the point*, to which the images γ_m^* of the arcs γ_m shrink, and let $\rho(p^*, p_0^*) < r_1^*$ and $\rho(p^*, p_0^*) < r_2^*$ $(0 < r_1^* < r_2^*)$ be two concentric balls containing no points of the image of the segment l. Since the arcs γ_m^* shrink to the point p_0^*, for any value $r_1^* > 0$ there is a number $m(r_1^*)$ such that for all $m > m(r_1^*)$ the arcs γ_m^* will lie in the ball $\rho(p^*, p_0^*) < r_1^*$. But then, for $m > m(r_1^*)$, all the curve families $\{c_m^*\}$ will intersect both spheres $\rho(p^*, p_0^*) = r_1^*$, $\rho(p^*, p_0^*) = r_2^*$, and by Grötzsch' s principles (1) the modulus $M\{c_m^*\}$ of each of these families is not greater than the modulus of the family of curves lying in the layer $r_1^* \leqslant \rho(p^*, p_0^*) \leqslant r_2^*$ and joining the spherical boundaries of the layer.

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* As a curve λ we take the homeomorphic image $\lambda: \ p = p(t)$ of the half-interval $[0 \leq t < 1]$, if the point $p = p(t)$ as $t \to 1$ tends to some point p_0, possibly infinitely remote; then we say that "the curve λ goes to the point p_0."

The modulus of the latter family is equal to $nV_n/[\ln(r_2^*/r_1^*)]^{n-1}$, where V_n is the volume of the n-dimensional unit ball.

Thus,

$$M\{c_m^*\} \leqslant nV_n/[\ln(r_2^*/r_1^*)]^{n-1} \quad \text{for } m > m(r_1^*). \tag{5}$$

Putting in estimate (5)

$$r_1^* = r_2^* \exp\left[-\frac{(2nV_n)^{1/(n-1)}Q}{\varepsilon^2}\right],$$

we obtain

$$M\{c_m^*\} \leqslant \frac{\varepsilon^{2(n-1)}}{2Q^{n-1}} \quad \text{for } m > m(r_1^*),$$

which contradicts relation (4). Lemma 4 is proved.

Definition 3. The collection (p, γ) consisting of a point p of the boundary of the domain D and a curve γ lying in D and going to the point p is called an **attainable boundary point of the domain** D. Two attainable boundary points (p, γ_1) and (p_2, γ_2) of the domain D are considered coincident if and only if the curves γ_1 and γ_2 are D-equivalent.

* Here p_0^* is regarded as a finite point; if necessary, this can be achieved by an inversion.

It is clear from the definition that an attainable boundary point (p, γ) can be identified with a class of D-equivalent curves going to the point p. Thus the attainable point (p, γ), like this entire class, is completely determined by specifying one curve of the class.

Theorem. *Under a homeomorphic Q-quasiconformal mapping of the ball R^n onto a domain D^*, to each attainable point (p_0^*, γ_0^*) of the boundary Γ^* of the domain D^* one can assign a definite point p_0 of the boundary sphere Γ in such a way that if a point p^* tends to p_0^* along any curve γ_0^* defining (p_0^*, γ_0^*), then the corresponding point p tends to p_0, and different attainable points of Γ^* correspond to different points of Γ. The set E of points of the sphere Γ corresponding to all attainable boundary points of the domain D^* is everywhere dense on every continuum $K \subset \Gamma$.*

Proof. Let $\gamma_0^* \subset D^*$ be a curve defining the attainable boundary point (p_0^*, γ_0^*), and let γ_0 be the preimage of this curve under the given Q-quasiconformal mapping. The curve γ_0 has on the sphere Γ only one limit point, for otherwise we could select in the ball R^n a sequence of arcs with endpoints converging to two distinct boundary points, whose images under a homeomorphic Q-quasiconformal mapping would contract to a point, which is impossible by

Lemma 4. Thus, if a point $p_0^* \in \gamma_0^*$ moves along γ_0^*, approaching p_0^*, then the corresponding point $p \in \gamma_0$, moving along γ_0, approaches one completely determined point $p_0 \in \Gamma$. We assign this point p_0 to the attainable boundary point (p_0^*, γ_0^*).

The established correspondence is correct, i.e., it does not depend on the choice of the curve defining the attainable boundary point. Indeed, if the curves γ_1^* and γ_2^* define one and the same attainable boundary point (p^*, γ^*), then in any neighborhood of the point p^* they can be joined by a curve lying in D^* and not leaving this neighborhood. If the curves γ_1, γ_2—the preimages of the curves γ_1^*, γ_2^*—defined different points p_1, p_2 on the sphere Γ, then the preimages of the sequence of curves joining γ_1^* with γ_2^* and contracting to the point p^* would be arcs with endpoints converging to distinct points $p_1 \in \Gamma,\ p_2 \in \Gamma$, which again contradicts Lemma 4.

If the curves γ_1^* and γ_2^* define different attainable boundary points, then without loss of generality one may assume that $\rho_{D^*}(\gamma_1^*, \gamma_2^*) > 0$. But then from Lemma $3'$ it follows at once that, under our correspondence, distinct attainable points of the boundary of the domain D^* correspond to distinct points on the sphere Γ.

It remains to show that the set E of points of the sphere Γ corresponding to all attainable boundary points of the domain D^* is everywhere dense on every continuum $K \in \Gamma$. Suppose that a continuum $K \subset \Gamma$ distinct from a point contains not a single point of the set E. This means that along no curve going to some point of the continuum K can our mapping have a limit; i.e., the length of the image of any such curve is infinite. But then the modulus of the image $\{c^*\} = T(\{c\})$ of the family $\{c\}$ of curves joining in R^n an arbitrary segment $\overline{p_1 p_2} \subset R^n$ with the continuum K is equal to zero, whereas the modulus of the family $\{c\}$ itself, by Lemma 1, is greater than zero. This contradicts the fact that the mapping $p^* \equiv T(p)$ is Q-quasiconformal. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.