E. R. Rozendorn

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Abstract

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E. R. Rozendorn

Properties of Asymptotic Lines on a Surface with Slowly Varying Negative Curvature

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We shall consider regular surfaces of negative Gaussian curvature K in three-dimensional Euclidean space E_3. As is known, on a surface of constant negative curvature the asymptotic lines form a Chebyshev net: in any asymptotic quadrilateral the lengths of opposite sides are pairwise equal ([2]). In the present note a generalization of the Chebyshev property of the asymptotic net is established for surfaces with negative Gaussian curvature differing little from a constant. It is proved that in this case, in an asymptotic quadrilateral, the sum of the lengths of two adjacent sides differs little from the sum of the lengths of the opposite pair of sides. The exact formulation is as follows:

Theorem 1. Let an arbitrary asymptotic quadrilateral be given on a surface with Gaussian curvature $K = -k^2 \leqslant -1$, the lengths of whose sides (numbered in the order of traversing the contour) are l_1, l_2, l_3, l_4. If $k \leqslant C_1$ and $|\operatorname{grad} k| \leqslant C_2 < 1$, then

$$\frac{1}{C}(l_1 + l_2) \leqslant l_3 + l_4 \leqslant C(l_1 + l_2), \tag{1}$$

where

$$C = \frac{C_1 + C_2}{1 - C_2}.$$

It follows from Theorem 1 that an expanding asymptotic quadrilateral on such a surface cannot open up until it reaches the boundary of the surface. Let us illustrate what has been said by the example of domains of a special kind. Suppose that on a surface of negative curvature two asymptotic lines passing through a point A are extended without bound in one direction, bounding a domain G homeomorphic to a half-plane, and the closure of G in the intrinsic metric contains no boundary points of the surface. A domain of this kind will be called an **asymptotic angle**. Asymptotic angles occur, for example, on a one-sheeted hyperboloid and on the universal covering of the pseudosphere. On a surface satisfying the conditions of Theorem 1, for example on the universal covering of the pseudosphere, the net of asymptotic lines inside any asymptotic angle is homeomorphic to the Cartesian net in a quadrant of the plane: each line of one family intersects all the lines of the other family. This property is also preserved under somewhat weaker restrictions imposed on the curvature of the surface.

Theorem 2. If on a surface with Gaussian curvature $K = -k^2$ there is an asymptotic angle G in which the inequalities

$$k \geqslant 1, \qquad |\operatorname{grad} k| \leqslant C_2, \qquad |\operatorname{grad}(\ln k)| \leqslant C_3, \qquad C_2 C_3 < 2, \tag{2}$$

hold, then the asymptotic net in G is homeomorphic to the Cartesian net in a quadrant of the plane.

Theorem 2 shows, in particular, that if there exists a complete surface satisfying inequalities (2), then on it the net of asymptotic lines as a whole is homeomorphic to the Cartesian net on the plane. However, the question of the existence in E_3 of such a surface remains open.

The example of the one-sheeted hyperboloid shows that Theorem 2 does not extend to arbitrary surfaces with $K < 0$.

It is known that for a surface of constant negative curvature the spherical image of the asymptotic lines also forms a Chebyshev net. In the case of variable negative curvature, under certain restrictions on the rate of its variation, the spherical images of asymptotic quadrilaterals have a property that may likewise be regarded as a generalization of the Chebyshev property. Namely, the following is true:

Theorem 3. *Let on a surface with curvature $K = -k^2 < 0$ there be an asymptotic quadrilateral D, whose spherical image $\overline{D}$ has area $\overline{\sigma}$, the length of one side $\bar{l}_1$, and the length of the opposite side $\bar{l}_3$. If in D the inequality*

$$\left|\operatorname{grad}\left(\frac{1}{k}\right)\right| \leqslant 2C_4$$

holds, then

$$|\bar{l}_3 - \bar{l}_1| \leqslant C_4 \overline{\sigma}.$$

Theorems 1 and 2 are based on the following lemma, which is also of independent interest:

Lemma. *Let on a surface with curvature $K = -k^2 \leqslant -1$ there be an asymptotic quadrilateral, whose area is σ, the length of one of its sides l_1, and the length of the opposite side l_3. If on the first side $k \leqslant C_1$, and in the whole quadrilateral $|\operatorname{grad} k| \leqslant C_2$, then*

$$l_3 \leqslant C_1 l_1 + \frac{1}{2} C_2 \sigma. \tag{3}$$

We shall prove the propositions formulated above, assuming for simplicity that the surfaces belong to the class C^3. (We note that the lemma and the theorems are also valid in the class C^2.) We shall consider the surface in asymptotic coordinates (u, v). Then the line element is written in the form $ds^2 = e^2 du^2 + 2eg\cos\omega\, du\, dv + g^2 dv^2$, where ω is the angle between the asymptotic directions at the given point. By subscripts we shall denote differentiation with respect to the coordinates u and v, and also with respect to the following variables: s_1 and s_2—with respect to the arc length of the lines $v = \text{const}$ and $u = \text{const}$, respectively; s_1^* and s_2^*—with respect to the arc length of the lines orthogonal to the lines $v = \text{const}$ and $u = \text{const}$, respectively.

To prove the lemma, consider the asymptotic quadrilateral $0 \leqslant u \leqslant t,\ 0 \leqslant v \leqslant b$, where $t \in [0, a]$, and a and b are fixed. The length of the side $u = t$ is

$$l(t) = \int_0^b g(t, v)\, dv,$$

the length of its spherical image is

$$\bar{l}(t) = \int_0^b k(t, v) g(t, v)\, dv,$$

and the area of the quadrilateral is

$$\sigma(t) = \int_0^a du \int_0^b eg\sin\omega\, dv.$$

We note that

$$\bar{l}(t) \leqslant l(t), \qquad \bar{l}(0) \leqslant C_1 l(0). \tag{4}$$

We use the following form of the Peterson-Codazzi equations in asymptotic coordinates, found in ([3]):

$$(ke)_v = \frac{1}{2} k_{s_1^*} eg\sin\omega, \qquad (kg)_u = -\frac{1}{2} k_{s_2^*} eg\sin\omega. \tag{5}$$

With the aid of (5) we find that

$$\bar{l}'(t) = \frac{1}{2}\int_0^b k_{s_2^*} eg\sin\omega\, dv \leqslant \frac{1}{2} C_2 \sigma'(t). \tag{6}$$

Integrating (6) from 0 to a and taking (4) into account, we obtain the required estimate

$$l(a) \leqslant C_1 l(0) + \frac{1}{2} C_2 \sigma(a).$$

Proof of Theorem 1. A. D. Aleksandrov proved [1] that if on a surface with curvature $K \leqslant -1$ there is a domain D, then its area $\sigma(D)$ and perimeter $p(D)$ are related by the inequality

$$\sigma(D) < p(D). \tag{7}$$

Applying (3) and (7) to the asymptotic quadrilateral, we obtain (1).

Proof of Theorem 2. Let the asymptotic lines forming the boundary G be taken as coordinate lines $v = 0,\ u \geqslant 0$ and $u = 0,\ v \geqslant 0$. It is necessary to prove that, for any positive a_1 and a_2, the asymptotic lines $u = a_1$ and $v = a_2$ intersect in G. Suppose the contrary: let, for some $a_1 > 0$ and $a_2 > 0$, the lines $u = a_1,\ v > 0$ and $v = a_2,\ u > 0$ go to infinity without intersecting. Denote by a_3 the sum of the lengths of the arcs $v = 0,\ 0 \leqslant u \leqslant a_1$ and $u = 0,\ 0 \leqslant v \leqslant a_2$, by a_4 the maximum of k on these arcs, and introduce two more constants: $a_5 = a_3(1 + a_4)$ and $a_6 = \frac{1}{2}C_3 a_5 + 2\pi$. For the geodesic curvature $\frac{1}{\rho_1}$ of the lines $v = \text{const}$ and $\frac{1}{\rho_2}$ of the lines $u = \text{const}$, we use the following formulas [4]:

$$\frac{1}{\rho_1} = -\omega_{s_1} + \frac{1}{2}(\ln k)_{s_2} \sin\omega, \qquad \frac{1}{\rho_2} = \omega_{s_2} - \frac{1}{2}(\ln k)_{s_1} \sin\omega. \tag{8}$$

For sufficiently small $t \in (0;1)$, the lines $u = a_1 t$ and $v = a_2 t$ intersect in G. Let D_t be the quadrilateral $0 \leqslant u \leqslant a_1 t,\ 0 \leqslant v \leqslant a_2 t$, its perimeter $p(D_t)$, area $\sigma(D_t)$, and sum of interior angles $\Omega(D_t)$. It follows from the lemma that

$$p(D_t) \leqslant C_2 \sigma(D_t) + a_5, \tag{9}$$

and by virtue of the assumption made above $\lim_{t\to 1} p(D_t) = +\infty$, therefore

$$\lim_{t\to 1} \sigma(D_t) = +\infty. \tag{10}$$

Applying the Gauss-Bonnet formula [2] to D_t, with the aid of (2), (8), and (9), we obtain the estimate

$$\begin{aligned} 0 = &\iint_{D_t} k^2 eg \sin\omega\, du\, dv + \frac{1}{2}\int_0^{a_1 t} (\ln k)_{s_2} e \sin\omega \Big|_{v=0}^{v=a_2 t} du \\ &+ \frac{1}{2}\int_0^{a_2 t} (\ln k)_{s_1} g \sin\omega \Big|_{u=0}^{u=a_1 t} dv + 2\pi - \Omega(D_t) \\ &\geqslant \sigma(D_t) - \frac{1}{2}C_3 p(D_t) \geqslant \left(1 - \frac{1}{2}C_2 C_3\right)\sigma(D_t) - a_6, \end{aligned}$$

which contradicts the relations (2) and (10).

Proof of Theorem 3 is carried out analogously to the proof of the lemma; it is only necessary to note that the line element of the spherical image is

$$d\bar{s}^2 = (ke)^2 du^2 - 2(ke)(kg)\cos\omega\, du\, dv + (kg)^2 dv^2$$

and that

$$\bar{\sigma} = \iint_D k^2 eg \sin\omega \, du \, dv.$$

Moscow State University
named after M. V. Lomonosov

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[4] N. V. Efimov, E. G. Poznyak, *DAN*, **137**, No. 1, 25 (1961).

Note: Figure translations are in progress. See original paper for figures.