MATHEMATICS

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Abstract

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CATEGORIES OF REPRESENTATIONS OF GROUPS AND THE PROBLEM OF CLASSIFYING IRREDUCIBLE REPRESENTATIONS

1. Let G be an arbitrary (discrete or continuous) group, and N a subgroup of it. Consider the set X of representations T_x $(x \in X)$ of the group G, induced by irreducible unitary representations of the subgroup N.* To each representation T_x we assign the ring R_x of all linear operators commuting with the operators of the representation. To each pair of representations T_{x_1}, T_{x_2}, acting respectively in the spaces H_1, H_2, we assign the linear family $R_{x_2 x_1}$ of all linear mappings from H_1 into H_2 that commute with the operators of the representations. The family of representations T_x, with the rings R_x and the linear spaces $R_{x_2 x_1}$, will be called the **category of representations of the group G, associated with the subgroup N**. The problem arises of describing this category. This problem is interesting for a number of reasons: its solution leads to a convenient classification of irreducible unitary representations of the group G (those entering the regular representation), gives a key to constructing special functions connected with the group G, etc.

In the present note we study the group G of unimodular matrices of the second order

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1,$$

with entries from a finite field K of order k (as is known, k is a power of a prime number p; we assume that $p \neq 2$). A description is given of the category of representations associated with the subgroup Z of matrices of the form

$$\xi_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

On the basis of this description, a convenient classification is given of all irreducible representations of the group G, as well as their effective construction (the latter problem is connected with the definition of Bessel functions over a finite field). At the end of the note similar results are formulated for the group of matrices over the field of real numbers.

2. Each irreducible representation of the subgroup Z is given by an additive character $\chi(t)$, $t \in K$ ($\chi(t_1 + t_2) = \chi(t_1)\chi(t_2)$). The representation T_χ of the group G induced by it is constructed in the space H_χ of functions $f(g)$ on G (with values in the field of complex numbers) satisfying the condition

$$f(\xi_t g) = \chi(t) f(g)$$

for any $g \in G$ and $t \in K$. The operator of the represen-

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* Let us recall the definition of an induced representation for the case of a unimodular subgroup N (for the general definition see, for example, (1)). Let $c(n)$ be a unitary representation of the subgroup N, acting in a space H with norm $\|h\|$. The representation $T(g)$ of the group G, induced by the representation $c(n)$, is constructed in the Hilbert space of functions $f(g)$ on G with values in H such that $f(ng) = c(n)f(g)$ for any $n \in N$ and

$$\int_{\widetilde{G}} \|f(\tilde{g})\|^2 \, d\tilde{g} < \infty,$$

where $\widetilde{G} = G/N$, $d\tilde{g}$ is an invariant measure on $\widetilde{G}$. The operator of the representation $T(g)$ is the shift operator:

$$T(g_0)f(g) = f(gg_0).$$

We note that in the case of Lie groups it is useful, besides unitary representations, also to consider nonunitary representations acting in certain naturally defined nuclear spaces.

has the form $T_\chi(g_0)f(g) = f(gg_0)$. Thus, in all there are k (k is the order of the field K) induced representations T_χ.

The representation T_χ can also be realized in the space of functions $f(x) \equiv f(x_1, x_2)$ on the "affine plane" $(x_1, x_2 \in K,\ (x_1, x_2) \neq (0,0))$; the set of such pairs (x_1, x_2) is identical with the set of cosets G/Z. In this realization the representation operator has the form

$$T_\chi(g)f(x) = f(xg)a(x,g),$$

where $xg = (\alpha x_1 + \gamma x_2,\ \beta x_1 + \delta x_2)$, and $a(x,g)$ is a fixed function satisfying the relations $a((0,1), \xi_t) = \chi(t)$, $a(x, g_1 g_2) = a(x, g_1) a(x g_1, g_2)$. Thus, in essence we are studying the category of representations of the group G connected with the affine plane.

3. Description of the rings R_χ and the spaces $R_{\chi_2\chi_1}$. The operators $A \in R_{\chi_2\chi_1}$ have the form

$$Af(g) = \frac{1}{k}\sum \varphi(gg_1^{-1})f(g_1)$$

(the sum is taken over all $g_1 \in G$), where $\varphi(g)$ is an arbitrary function on G satisfying, for any $g \in G,\ t_1, t_2 \in K$, the relation

$$\varphi(\xi_{t_2} g \xi_{t_1}) = \chi_2(t_2)\varphi(g)\chi_1(t_1). \tag{1}$$

Hence, for $\chi_1 = \chi_2 = \chi$, we obtain that R_χ is isomorphic to the ring of functions $\varphi(g)$ satisfying relation (1), with multiplication law

$$\varphi_1(g) * \varphi_2(g) = \frac{1}{k}\sum_{g_1} \varphi_1(gg_1^{-1})\varphi_2(g_1) * .$$

In particular, the ring R_1, corresponding to the character $\chi \equiv 1$, is isomorphic to the ring of functions constant on the double cosets with respect to the subgroup Z. We first describe this ring R_1.

Let A_λ be the characteristic function of the double coset with respect to Z with representative

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix},$$

and B_λ the characteristic function of the coset with representative

$$\begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix}.$$

Then A_λ, B_λ form a basis in the linear space R_1 (and hence the dimension of R_1 is $2k-2$). The multiplication operation in R_1 is given by the formulas

$$A_{\lambda_1}A_{\lambda_2} = A_{\lambda_1\lambda_2}, \quad A_{\lambda_1}B_{\lambda_2} = B_{\lambda_1\lambda_2}, \quad B_{\lambda_1}A_{\lambda_2} = B_{\lambda_1\lambda_2^{-1}}, \quad B_{\lambda_1}B_{\lambda_2} = \sum_\lambda B_\lambda + kA_{-\lambda_1\lambda_2^{-1}}.$$

From these formulas it is easy to find that the center of the ring R_1 consists of elements of the form

$$A = \alpha_1 A_1 + \alpha_{-1}A_{-1} + \sum_{\lambda'} \beta_\lambda (A_\lambda + A_{\lambda^{-1}}) + \gamma' \sum_{\lambda'} B_\lambda + \gamma'' \sum_{\lambda''} B_\lambda,$$

where λ' runs through the set of all squares, and λ'' through the complementary set (and hence the dimension of the center is $\frac{1}{2}(k+5)$). Further, the elements A of the ring satisfying the condition $A'A = c_{A'}A$ for every element A' of the ring have, up to a scalar factor, the form

$$A = \pm\sqrt{k\pi(-1)}\sum \pi(\lambda)A_\lambda + \sum \pi(\lambda)B_\lambda,$$

where $\pi(\lambda') = 1,\ \pi(\lambda'') = -1$ (the notation λ', λ'' is the same as above), or the form

$$A = \sum_\lambda (A_\lambda + B_\lambda), \qquad A = \sum_\lambda (-kA_\lambda + B_\lambda).$$

From the formulated results it follows immediately: the ring R_1 is a direct sum of $\frac{1}{2}(k+5)$ full matrix rings, of which 4 summands are matrices of order 1, and the remaining $\frac{1}{2}(k-3)$ summands are matrices of order 2.

Now let $\chi \not\equiv 1$. Then the elements of the ring R_χ—the functions $\varphi(g)$—are equal to zero on double cosets with respect to Z having representatives

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \quad \lambda \neq \pm 1.$$

We define in the space R_χ a basis A_1, A_{-1}, B_λ. Each of the functions A_1, A_{-1}, B_λ is determined by the following conditions: it is equal to 1 respectively on the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix}$$

and is nonzero only on the double coset with respect to Z containing this matrix. The multiplication operation in $R_\chi,\ \chi \not\equiv 1$, is given by formu-

* We note that the rings R_χ are semisimple, i.e., decompose into a direct sum of full matrix rings (this follows from the complete reducibility of representations of a finite group).

by

$$A_{-1}A_{-1} = A_1, \quad B_\lambda A_{-1} = A_{-1}B_\lambda = B_{-\lambda}, \quad B_{\lambda_1}B_{\lambda_2} = \sum_\lambda \chi\left(\frac{\lambda}{\lambda_1\lambda_2} + \frac{\lambda_1}{\lambda\lambda_2} + \frac{\lambda_2}{\lambda\lambda_1}\right) B_\lambda + k\delta_{\lambda_1+\lambda_2}A_1 + k\delta_{\lambda_1-\lambda_2}A$$

A_1 is the identity of the ring. (Here $\delta_t = 1$ for $t = 0$, $\delta_t = 0$ for $t \neq 0$.) Hence we conclude that for $\chi \neq 1$ the ring R_χ is commutative and has dimension $k+1$.

Taking into account the relations (1), it is easy to determine the dimensions of the spaces $R_{\chi_2\chi_1},\ \chi_1 \neq \chi_2$. If the characters $\chi_1(t)$ and $\chi_2(t)$ are related by

$$\chi_2(t) = \chi_1(\lambda^2 t)$$

for some $\lambda \neq 0$, then the dimension of $R_{\chi_2\chi_1}$ is equal to $k+1$; otherwise the dimension of $R_{\chi_2\chi_1}$ is equal to $k-1$.

4. **Classification of irreducible representations of the group G.** Let us divide the representations T_χ, $\chi \neq 1$, into two classes; we assign T_{χ_1} and T_{χ_2} to the same class if

$$\chi_2(t) = \chi_1(\lambda^2 t)$$

for some $\lambda \neq 0$. On the basis of §3 we obtain: the representation T_χ, $\chi \neq 1$, is the direct sum of $k+1$ pairwise inequivalent representations. If the representations T_{χ_1} and T_{χ_2} belong to the same class, then they are equivalent; but if T_{χ_1} and T_{χ_2} belong to different classes, then they have $k-1$ common irreducible summands. The representation T_1, corresponding to the character $\chi = 1$, contains 4 irreducible representations with multiplicity 1 and $\frac{1}{2}(k-3)$ irreducible representations with multiplicity 2. The representations T_χ, $\chi \neq 1$, and T_1 have $\frac{1}{2}(k+1)$ common irreducible components; of these, two occur in T_1 with multiplicity 1, and the remaining ones with multiplicity 2. We can now classify all irreducible representations of the group G according to the multiplicities with which they occur in the representations T_χ. (We note that the regular representation of the group G is the direct sum of the representations T_χ; hence every irreducible representation of the group G is contained in at least one of the representations T_χ.)

We obtain the following types of irreducible representations: 1) representations of the "principal" series; they occur in all T_χ, and in T_1 they occur with multiplicity 2; the number of such representations is $\frac{1}{2}(k-3)$, the dimension of each is $k+1$. 2) Representations of the "analytic" series; they occur in all T_χ except T_1; the number of such representations is $\frac{1}{2}(k-1)$, the dimension of each is $k-1$. 3) One representation of dimension k; it occurs in all T_χ, and in T_1 it occurs with multiplicity 1. 4) Two representations of dimension $\frac{1}{2}(k+1)$; each of them occurs in the representations T_χ, $\chi \neq 1$, belonging to only one of the two classes; in T_1 they occur with multiplicity 1. 5) Two representations of dimension $\frac{1}{2}(k-1)$. Each of them occurs in the representation T_χ, $\chi \neq 1$, belonging to only one of the two classes, and does not occur in T_1. 6) The identity representation; it is contained only in T_1.

5. All irreducible representations of the group G can be obtained by decomposing the spaces H_χ into irreducible subspaces. In the case $\chi \equiv 1$, the irreducible subspaces are the subspaces of homogeneous functions of a given degree of homogeneity π ($\pi(t)$ is a multiplicative character):

$$f(tx_1, tx_2) = \pi(t) f(x_1, x_2)$$

for any $t \neq 0$*. We note that in these subspaces only one half of all irreducible representations is realized. Here a description will be given of all irreducible subspaces of the space H_χ, $\chi \neq 1$. It is interesting that the analogues of homogeneous functions for $\chi \neq 1$ are Bessel functions.

Consider functions $\varphi(g) \in R_\chi$ satisfying the condition

$$\psi(g) * \varphi(g) = c_\psi \varphi(g) \tag{2}$$

for every $\psi \in R_\chi$; c_ψ is a complex number (the mapping $\psi \mapsto c_\psi$ is a homomorphism of the ring R_χ). The functions $\varphi(g)$ will be normalized by the condition $\varphi(e) = 1$, where e is the identity of the group. Between such functions $\varphi(g)$ and irreducible subspaces of the space H_χ, $\chi \neq 1$, there exist–

* Not counting the special cases when $\pi(t) \equiv 1$ or $\pi(t) = \pm 1$. In each of these cases the space of homogeneous functions is the sum of two irreducible subspaces.

there is a one-to-one correspondence. The irreducible subspace corresponding to the function $\varphi(g)$ consists of all functions representable in the form $\varphi(g) * f(g)$, where $f(g)$ ranges over H_χ. The problem is to describe all functions $\varphi(g)$ satisfying condition (2). This description is given below.

In view of (1), it is enough for us to know the values of the functions $\varphi(g)$ only on the matrices $\pm e$ and

$$\begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix}.$$

Denote by $\varphi(\pm e)$ and $\varphi(\lambda)$ the values of the function $\varphi(g)$ on these matrices. Then for $\varphi(\lambda)$ we obtain the following functional equation:

$$k\varphi(\lambda_1)\varphi(\lambda_2) = \varphi(-e)\sum_\lambda \chi\left(\frac{\lambda}{\lambda_1\lambda_2} + \frac{\lambda_1}{\lambda\lambda_2} + \frac{\lambda_2}{\lambda\lambda_1}\right)\varphi(\lambda) +$$

$$+\delta_{\lambda_1-\lambda_2}\varphi(-e) + \delta_{\lambda_1+\lambda_2}\varphi(e) \qquad (3)$$

for any $\lambda_1, \lambda_2 \neq 0$; $\varphi(-e)\varphi(\lambda) = \varphi(-\lambda)$; $\varphi(-e) = \pm\varphi(e)$, where $\varphi(e) = 1$. There are two classes of solutions of equation (3). The functions of the first class are given by the formula

$$\varphi(\lambda) = J_\pi(\lambda;\chi) \equiv \frac{1}{k}\sum_t \chi\left(-\lambda^{-1}(t+t^{-1})\right)\pi(t); \qquad \varphi(-e) = \pi(-1),$$

where $\pi(t)$ is a multiplicative character on K; the summation is over all $t \neq 0$ from K. To describe the second class of solutions of equation (3), extend the field K by adjoining to it one square root. In the resulting extension consider the set K_1 of elements "equal in modulus to one" (i.e., such that the product of an element by its conjugate is equal to 1). This set is a multiplicative group of order $k+1$. The functions of the second class are given by the formula

$$\varphi(\lambda) = K_\pi(\lambda;\chi) = -\frac{1}{k}\sum_t \chi\left(-\lambda^{-1}(t+t^{-1})\right)\pi(t); \qquad \varphi(-e) = \pi(-1),$$

where $\pi(t)$ is a multiplicative character on K_1, $\pi \neq 1$; the summation is over all $t \neq 0$ from K_1.* The functions $J_\pi(\lambda;\chi)$ and $K_\pi(\lambda;\chi)$ are naturally called **Bessel functions over the finite field** K. (The formulas for $J_\pi(\lambda;\chi)$ and $K_\pi(\lambda;\chi)$ are analogous to the formulas of the integral representation of ordinary Bessel functions.)

6. Let us formulate analogous results for the group G_B of unimodular matrices of the second order over the field of real numbers. It can be shown that also for this group the ring R_χ, $\chi(t) = e^{i\sigma t}$, is commutative for $\chi \neq 1$. In

this case the decomposition of the representation T_χ into irreducible representations contains all irreducible representations of the principal continuous series and half of the irreducible representations of the discrete series. (The other half of the representations of the discrete series enters into the decomposition of the representation $T_{\bar{\chi}}$, $\bar{\chi}(t) = e^{-i\sigma t}$.) The realization of irreducible unitary representations (of the principal series) can be carried out in the same way as in § 5 for the case of a finite field. The determination of functions $\varphi(g)$ satisfying condition (2) is easily reduced to solving a differential equation of the form

$$\lambda^2\varphi''(\lambda) + 3\lambda\varphi'(\lambda) + \left(4\sigma^2/\lambda^2 - \nu\right)\varphi(\lambda) = 0;$$

the solutions of this equation are the functions $\lambda^{-1}J_\alpha(2\sigma/\lambda)$, $\alpha = \pm\sqrt{1+\nu}$, where $J_\alpha(x)$ is the Bessel function.

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* Note that the functions of the first class correspond to irreducible representations contained in $\widetilde{T}_1$, while the functions of the second class correspond to those not contained in T_1.

Note: Figure translations are in progress. See original paper for figures.