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Abstract

Full Text

MATHEMATICS

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FINDING MINIMAL COLORINGS OF THE VERTICES OF A GRAPH BY MEANS OF BOOLEAN POWERS OF THE ADJACENCY MATRIX

(Presented by Academician S. L. Sobolev, June 14, 1962)

Let G be a finite undirected graph (without loops and parallel edges). Orienting all its edges in an arbitrary way, we obtain an antisymmetric graph $\vec{G} = (X, \Gamma)$. By $\gamma(G)$ we denote the chromatic number of the graph G, and by $k(G)$ the greatest of those integers m such that, under any orientation, the corresponding graph $\vec{G}$ contains at least one path of length m.

Theorem 1. $\gamma(G) = k(G) + 1$.

Proof. 1. Let $\gamma(G) = m$. Color the vertices of the graph G with the colors $1, 2, \dots, m$ (so that adjacent vertices are not colored alike), and orient each edge in the direction from the vertex with the larger color number to the vertex with the smaller color number. It is clear that in the graph G thus obtained there are no paths of length m, i.e. $k(G) \le m - 1$, or

$$\gamma(G) \ge k(G) + 1.$$

2. Let $k(G) = l$. Consider the graph $\vec{G} = (X, \Gamma)$, obtained from G by such an orientation of the edges that there are no paths of length $l + 1$. Put

$$M_1 = \{x/\Gamma x = \emptyset\}; \qquad M_i = \{x/\Gamma^{i-1} x \subset M_1\} \quad (i > 1),$$

i.e. M_1 denotes the set of all terminal vertices of the graph $\vec{G}$, and M_i, for $i > 1$, the set of those of its vertices from which one can reach some terminal vertex along a path of length $i - 1$. Further, let

$$S_i = M_i \setminus \bigcup_{j>i} M_j,$$

i.e. S_i is the set of those vertices of the graph $\vec{G}$ from which one can reach a terminal vertex along a path of length $i - 1$, but cannot reach terminal vertices along any path of greater length. It is clear that $S_i \ne \emptyset$ $(i = 1, 2, \dots, l + 1)$;

$$S_i \cap S_j = \emptyset \quad (i \neq j) \quad \text{and} \quad \bigcup_{i=1}^{l+1} S_i = X.$$

No two vertices of one and the same set S_i are adjacent, since from $x \in S_i,\ y \in S_i,\ y \in \Gamma x$ it would follow that $x \in S_{i+1}$. Coloring all vertices of the set S_i with the i-th color, we obtain a proper coloring of the graph G by means of $l+1$ colors. Consequently, $\gamma(G) \leq l+1$, i.e.

$$\gamma(G) \leq k(G) + 1.$$

The theorem is proved.

Let $A = \|a_{ij}\|$ be the adjacency matrix (for edges) of the graph G, and $B = \|b_{ij}\|$ the adjacency matrix (for arcs) of the graph $\vec{G}$.

We shall consider the quantities a_{ij} and b_{ij} as elements of the Boolean algebra $\mathfrak{B}$, taking the values 0 and 1.

Each way of orienting the edges of the graph G can be specified by a system of elements $y_{ij} \in \mathfrak{B}$, where

$$y_{ij} = \begin{cases} 1, & \text{if } x_j \in \Gamma x_i \text{ and } j > i, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$b_{ij} = \begin{cases} a_{ij} y_{ij}, & \text{for } j > i, \\ a_{ij} \bar{y}_{ij}, & \text{for } j < i, \\ 0, & \text{for } j = i. \end{cases}$$

As is known ([1], Ch. 14), the element $b_{ij}^{(\alpha)}$ of the matrix B^α is equal to 1 if and only if some path of length α leads from the vertex x_i to the vertex x_j. Consequently, the existence in the graph $\vec{G}$ of paths of length α is equivalent to the function

$$Q_\alpha(\{y_{ij}\}) = \sum_{i,j=1}^{|X|} b_{ij}^{(\alpha)}$$

taking the value 1 under the distribution of the values of the arguments y_{ij} that corresponds to the given orientation of the edges of the graph.

From Theorem 1 there follows immediately

Theorem 2. $\gamma(G)$ is equal to the smallest number $\alpha+1$ for which $Q_{\alpha+1}(\{y_{ij}\}) \not\equiv 1$.

The following algorithm for finding the chromatic number and obtaining certain minimal colorings of a graph G, given by the adjacency matrix A for its edges, is based on these results.

Form the matrices $B(\{y_{ij}\}) = \|b_{ij}\|$, where the y_{ij} are variables, and the function

$$Q_1 = \sum_{i,j=1}^{|X|} b_{ij};$$

if $Q_1 \not\equiv 1$, then $\gamma(G) = 1$ (the trivial case). If $Q_1 \equiv 1$, then form the matrix B^2 and the function Q_2; if $Q_2 \not\equiv 1$, we have $\gamma(G) = 2$. If $Q_2 \equiv 1$, then form B^3 and Q_3, etc. After a finite number of steps we find such an α that $Q_\alpha \equiv 1$, $Q_{\alpha+1} \not\equiv 1$. Then $\gamma(G) = \alpha + 1$.

To obtain some coloring of the vertices of the graph G by means of $\alpha+1$ colors, it is enough to choose such a system of values $\{y_{ij}^0\}$ for which $Q_{\alpha+1} = 0$, and then to find successively the sets $S_{\alpha+1}, S_\alpha, \dots, S_1$ (see the proof of Theorem 1) as follows: if $S_{\alpha+1}, S_\alpha, \dots, S_{m+1}$ $(m > 1)$ have already been constructed, then in the matrix $B^{m-1}(\{y_{ij}^0\})$ replace by zeros all elements of those rows and columns that correspond to vertices from $\bigcup_{p=m+1}^{\alpha+1} S_p$; the set S_m is composed of those vertices x_i to which, in the resulting matrix, nonzero columns correspond; $S_1 = X \setminus \bigcup_{q>1} S_q$.

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CITED LITERATURE

1. K. Berge, *Théorie des graphes et ses applications*, Paris, 1958.

Note: Figure translations are in progress. See original paper for figures.