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Abstract

Full Text

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A DIOPHANTINE EQUATION WITH A MATRIX EXPONENTIAL FUNCTION

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In paper ([1]) M. P. Mineev gave a new proof of the Fortet-Kac theorem ([2]). M. P. Mineev relied on the asymptotic formula proved by him for the number of solutions of the Diophantine equation

$$m_1 g^{x_1} + \dots + m_k g^{x_k} = n_1 g^{y_1} + \dots + n_k g^{y_k}, \tag{1}$$

where $m_1, \dots, m_k, n_1, \dots, n_k$ are positive rational integers, g is an integer ≥ 2, in nonnegative integers of the interval $[0, p-1]$ as p grows.

In the present note an asymptotic formula with a remainder term is obtained for the number of solutions of the matrix Diophantine equation

$$\widetilde{m}_1 \psi(A^{x_1}) + \dots + \widetilde{m}_a \psi(A^{x_a}) = \tilde{n}_1 \psi(A^{y_1}) + \dots + \tilde{n}_b \psi(A^{y_b}) \tag{2}$$

in nonnegative integers $x_1, \dots, y_b$ of the interval $[0, p-1]$, where A is a nonsingular integral matrix of order n, among whose eigenvalues there is no root of unity; $\psi(x)$ is an arbitrary polynomial with integral coefficients, not identically constant; $\widetilde{m}_1, \dots, \widetilde{m}_a, \tilde{n}_1, \dots, \tilde{n}_b$ are n-dimensional integral vectors.

Notation. If $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n)$, then by the symbol $\{\tilde{\alpha}\}$ we shall mean the vector $(\{\alpha_1\}, \dots, \{\alpha_n\})$, where $\{\alpha_j\}$ is the fractional part of α_j. A is a nonsingular integral matrix of order n, among whose eigenvalues there is no root of unity. $\psi(x)$ is an arbitrary polynomial with integral coefficients, not identically constant. $N_p(\Delta)$ is the number of fractional parts $\{\tilde{\alpha} A^x\}$, $x = 0, 1, \dots, p-1$, that have fallen into the hypercube Δ. $\varepsilon(t)$ is an arbitrary nonnegative function tending to zero as slowly as desired as $t \to 0$. $|\Delta|$ is the volume of the hypercube Δ, mes is Lebesgue measure, π is the hypercube $(0 \leq x_1 \leq 1, \dots, 0 \leq x_n \leq 1)$.

On the basis of the formula proved for the number of solutions of equation (2), the following theorems have been proved:

Theorem. If, for some $c > 0$,

$$\overline{\lim}_{p \to \infty} \frac{N_p(\Delta)}{p} \leq c |\Delta|^{1-\varepsilon(|\Delta|)} \tag{3}$$

for every n-dimensional hypercube Δ lying in the hypercube π, then the sequence $\{\tilde{\alpha}A^x\}$, $x = 0, 1, ...,$ is uniformly distributed in the hypercube π.

The condition stated in the theorem cannot be improved in the sense that $\varepsilon(t)$ cannot be replaced by a constant function different from zero in any interval $(0, t)$, as was shown in paper ([3]) for a special case.

A weaker theorem for matrices, in the sense of the requirement of condition (3), was proved by A. M. Polosuev ([4]) and I. Cigler ([5]).

Central limit theorem. Let $f(\tilde{x}) = f(x_1, \dots, x_n)$ be a real function periodic in each argument

with period 1, a function with Fourier coefficients $a_{m_1,\dots,m_n} = a_{\widetilde{m}}$ such that

$$\sum \|\widetilde{m}\|^{\varepsilon} |a_{\widetilde{m}}|^2 < \infty$$

for some $\varepsilon > 0$, where $\|\widetilde{m}\|$ is the usual norm of the vector, and $a_{\tilde{0}} = 0$. Then:

1) the limit

$$\lim_{p\to\infty} \int_{\pi} \left(\frac{1}{\sqrt{p}} \sum_{t=0}^{p-1} f(\tilde{\alpha}\psi(A^t)) \right)^2 d\tilde{\alpha}$$

exists; denote it by σ^2.

2)

$$\operatorname*{mes}_{\tilde{\alpha}\in\pi} E \left\{ \frac{1}{\sqrt{p}} \sum_{t=0}^{p-1} f(\tilde{\alpha}\psi(A^t)) < \lambda \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda/\sigma} e^{-t^2/2}\, dt + O\left(c\, \frac{\sqrt{\log\log p}}{\sqrt{\log p}} \right),$$

where

$$c = \begin{cases} 1, & \text{if } \sigma = 0, \\ \dfrac{\|f\|_{\varepsilon}}{\sigma}, & \text{if } \sigma \neq 0, \end{cases} \qquad \|f\|_{\varepsilon} = \left[\sum \|\widetilde{m}\|^{\varepsilon} |a_{\widetilde{m}}|^2 \right]^{1/2};$$

the O-term contains quantities depending on the matrix A, the polynomial $\psi(x)$, and ε.

Remark 1. If $\sigma = 0$, then by the symbol λ/σ we mean $+\infty$ for $\lambda > 0$ and $-\infty$ for $\lambda < 0$.

Remark 2. If $\psi(x)$ has two nonzero coefficients, then $\sigma = \|f\|$.

Theorem. Let $f(\tilde{x})$ be a complex function, periodic in each argument with period 1; suppose that for some $\varepsilon > 0$

$$\sum \|\widetilde{m}\|^{\varepsilon} |a_{\widetilde{m}}|^2 < \infty,$$

where $a_{\widetilde{m}}$ are the Fourier coefficients of the function $f(\tilde{x})$ on the hypercube π, and $a_{\tilde{0}} = 0$. Then:

1) The limits

$$\lim_{p\to\infty} \int_{\pi} \left(\frac{1}{\sqrt{p}} \sum_{t=0}^{p-1} \operatorname{Re} f(\tilde{x}\psi(A^t)) \right)^2 dx, \qquad \lim_{p\to\infty} \int_{\pi} \left(\frac{1}{\sqrt{p}} \sum_{t=0}^{p-1} \operatorname{Im} f(\tilde{x}\psi(A^t)) \right)^2 d\tilde{x},$$

$$\lim_{p\to\infty} \int_{\pi} \left(\frac{1}{\sqrt{p}} \sum_{t=0}^{p-1} \operatorname{Re} f(\tilde{x}\psi(A^t)) \right) \left(\frac{1}{\sqrt{p}} \sum_{t=0}^{p-1} \operatorname{Im} f(\tilde{x}\psi(A^t)) \right) d\tilde{x}$$

exist; denote them, respectively, by $\sigma_{20}, \sigma_{02}, \sigma_{11}$.

2)

$$\operatorname*{mes}_{\tilde{x}\in\pi} E \left\{ \frac{1}{\sqrt{p}} \left| \sum_{t=0}^{p-1} f(\tilde{x}\psi(A^t)) \right| < \lambda \right\}$$

$$= 1 - \frac{\sqrt{M}}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2}\lambda^2 \frac{\delta(\theta)}{M}} \frac{d\theta}{\delta(\theta)} + O\left(\frac{(\log\log p)^{3/2}}{\sqrt{\log p}} \right) \quad \text{if } M > 0,$$

where

$$M = \sigma_{20}\sigma_{02} - \sigma_{11}^2, \qquad \delta(\theta) = \sigma_{20}\cos^2\theta - 2\sigma_{11}\cos\theta\sin\theta + \sigma_{02}\sin^2\theta.$$

A special case of this theorem without indicating the remainder term was obtained by M. P. Mineev.

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Note: Figure translations are in progress. See original paper for figures.