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GENERAL BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC SYSTEMS IN A MULTIDIMENSIONAL DOMAIN

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In this note, for systems elliptic in the sense of I. G. Petrovskii, results are established analogous to those obtained in (^{1}), where the case of a single elliptic equation was considered.

Let G be a bounded domain in the Euclidean space R^n of points $x = (x^1, \dots, x^n)$. We assume that its boundary Γ is an $(n-1)$-dimensional infinitely smooth* surface admitting local "straightening" by means of coordinate transformations (see, for example, ($^{2-4}$)). Consider in the domain G the operator

$$Au = A(x, D)u(x). \tag{1}$$

Here $u(x)$ is a column of p functions; $A(x, D)$ is a square matrix of order p, consisting of linear partial differential operators of order s with complex coefficients infinitely smooth in $\overline{G}$; $D = (D_1, \dots, D_n)$, where $D_j = -i\,\partial/\partial x^j$. We assume the operator A to be elliptic:

$$\det A_0(x, \xi) \neq 0 \quad \text{for } x \in \overline{G} \text{ and } \xi = (\xi_1, \dots, \xi_n) \neq 0, \tag{2}$$

where $A_0(x, D)$ is the principal part of A (homogeneous in D of degree s); the ξ_j are real. For $n > 2$, it follows from (2) that ps is even (^{5}); for $n = 2$ this is assumed additionally.

On Γ let us prescribe the boundary operator

$$Bu = B(x, D)u(x)\big|_{\Gamma}. \tag{3}$$

Here $B(x, D)$ is a matrix with $r = ps/2$ rows and p columns, consisting of linear partial differential operators; the coefficients in these operators will be, as in (^{1}) (see also (^{6})), singular integral operators on Γ with infinitely smooth complex kernels and free terms (see (^{7,8})). In particular, if all kernels are equal to 0, then B will be an ordinary differential operator. Let m_k be the highest order of differentiation in the k-th row of the matrix B. Denote by $B_0(x, D)$ the principal part of B, obtained by discarding, in the k-th row, terms of order $< m_k$ $(k = 1, \dots, r)$.

We shall subject the operators A and B on Γ to the condition of Ya. B. Lopatinskii (L), which we now formulate (cf. (^{5,9,10})). Take an arbitrary point M on Γ. In order not to introduce complicated notation, suppose that the origin of the coordinate system x is at M and that the axis x^n is directed along the inward normal to Γ at M. On functions defined in the half-space $R^n+ : \;\; x^n > 0$, consider the operators

$$A_0(0, D)u(x), \qquad B_0(0, D)u(x)\big|_{x^n=0}. \tag{4}$$

The first of them is obtained from $A_0(x, D)$ by freezing the coefficients at $x = 0$. In the second, the coefficients are singular integral operators on the hyperplane $x^n = 0$. Their symbols (see (^{7,8})) are obtained by freezing at $x = 0$ the symbols of the corresponding integral operators entering into $B_0(x, D)$.

In (4) perform the Fourier transform $\dot{F}$ with respect to the variables $x^1, \dots, x^{n-1}$, and consider, for arbitrarily fixed $\dot{\xi} = (\xi_1, \dots, \xi_{n-1}, 0)$, the boundary-value problem on the half-line:

$$A_0(0, \dot{\xi}, D_n)v(x^n) = 0 \qquad (x^n > 0); \tag{5}$$

$$B_0(0, \dot{\xi}, D_n)v(x^n) = h \qquad (x^n = 0). \tag{6}$$

Here h is a numerical column of height r. Following $^{(9)}$, we shall call a solution of system (5) stable if it tends to 0 as $x^n \to +\infty$. For $n > 2$ it is proved that the dimension of the space $\mathfrak{M}$ of stable solutions is equal to r $^{(5)}$; for $n = 2$ we shall assume this additionally. An operator A with this property is called properly elliptic.

The condition (L) at the point M of the boundary Γ consists in the fact that for every $\dot{\xi} \neq 0$ problem (5), (6) is uniquely solvable in $\mathfrak{M}$.

For what follows it is necessary to write condition (L) in explicit form. To this end take some stable basis $\omega_1(x^n), \dots, \omega_r(x^n)$, i.e. a basis in the space $\mathfrak{M}$, and put

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* All assumptions on smoothness may be weakened.

$$v(x^n) = a_1\omega_1(x^n) + \dots + a_r\omega_r(x^n), \tag{7}$$

where a_j are numerical coefficients. Denote by ω the matrix composed of the columns $\omega_1, \dots, \omega_r$. Substituting (7) into (6), we obtain a system of r linear algebraic equations with respect to $a_1, \dots, a_r$. The condition for its unique solvability is that the matrix

$$B_0\omega(\dot\xi) = B_0(0, \dot\xi, D_n)\,\omega(x^n)\big|_{x^n=0} \qquad (\dot\xi \neq 0) \tag{8}$$

must be nonsingular. This is precisely condition (L). Obviously, it does not depend on the choice of the stable basis.

Put $\mathfrak{A} = (A, B)$. We shall call the operator $\mathfrak{A}$ **elliptic** if A is properly elliptic and if A and B are connected on Γ by condition (L).

Let k be an integer ≥ 0. Denote by $H_k(G)$ the direct product of p (scalar) spaces of S. L. Sobolev $W_2^{(k)}(G)$. If $k \geq \max m_j + 1$, then denote by $H_{k-m-1/2}(\Gamma)$ the direct product of the spaces of L. N. Slobodetskii $^{(11)}$

$$W_2^{(k-m_j-1/2)}(\Gamma) \qquad (j = 1, \dots, r).$$

We agree to denote the norms in these vector spaces by $\|\ \|$ with the corresponding indices. In this case the square of the norm of a vector is equal to the sum of the squares of the norms of its components.

In what follows l will be an integer $\geq l_0 = \max(s, m_j + 1)$. By $H_l(G, \Gamma)$ denote the direct product of the spaces $H_{l-s}(G)$ and $H_{l-m-1/2}(\Gamma)$. We introduce analogous notation for $R^n_+, \dot R^n_+$ instead of G, Γ.

The operator $\mathfrak{A}$ acts from $H_l(G)$ into $H_l(G, \Gamma)$ and is bounded.

Theorem 1. *Ellipticity of the operator $\mathfrak{A}$ is equivalent to each of the following two conditions:*

I. *If $u \in H_{l_0}(G)$, $Au \in H_{l-s}(G)$, and $Bu \in H_{l-m-1/2}(\Gamma)$, then $u \in H_l(G)$ and the a priori estimate holds*

$$\|u\|_l \leq C(\|Au\|_{l-s} + \|Bu\|_{l-m-1/2} + \|u\|_0), \tag{9}$$

where C is a constant independent of $u(x)$.

II. *$\mathfrak{A}$ is a Φ-operator from $H_l(G)$ into $H_l(G, \Gamma)$.*

The latter means that the equation $\mathfrak{A}u = 0$ has a finite number α of linearly independent solutions in $H_l(G)$, that the range of the operator $\mathfrak{A}$ is closed in $H_l(G, \Gamma)$, and that the quotient space $H_l(G, \Gamma)/\mathfrak{A}H_l(G)$ has finite dimension β $^{(12)}$. If the operator $\mathfrak{A}$ is elliptic, then from I it follows that the numbers α and

β do not depend on l. The difference $\varkappa(\mathfrak{A}) = \alpha - \beta$ is called the index of the operator $\mathfrak{A}$.

For lack of space there is no possibility here to dwell in detail on the prehistory of Theorem 1 (see ($^{2-5,9,10,13-16}$), where further references may be found). In the cited works differential boundary conditions were considered, and Theorem 1 had not yet been established for them in full generality. In (^{1}) it is proved for integro-differential boundary conditions for $p = 1$.

Let us outline the derivation of assertion II from the ellipticity of the operator $\mathfrak{A}$. First one constructs a stable basis $\Omega_1, \dots, \Omega_r$ for (5), where Ω_j satisfies condition (6) with a column h all of whose elements are equal to 0 except the j-th, which is equal to 1. Denote by Ω the matrix whose columns are the Ω_j. The inverse Fourier transforms of its elements are the "Poisson kernels" (cf. (^{3})). For $p = 1$ there are explicit formulas (^{3}) for Ω_j. For $p \geqslant 1$ one can show (cf. (^{3,5})) that the following is true.

Lemma 1. *The matrix Ω admits the representation*

$$\Omega(\xi, x^n) = \int_{\gamma(\xi)} e^{ix^n\lambda} A_0^{-1}(0, \xi, \lambda) N(\xi, \lambda)\, d\lambda. \tag{10}$$

Here $\gamma(\xi)$ is a contour in the half-plane $\operatorname{Im}\lambda > 0$, encircling the zeros of $\det A_0(0, \xi, \lambda)$ lying there, and N is a matrix with p rows and r columns, the elements of whose j-th column, infinitely differentiable for $\xi \neq 0$, are polynomials in λ and are positively homogeneous in (ξ, λ) of degree $m_j - s - 1$ $(j = 1, \dots, r)$.

We shall call a **regularizer** for the operator $\mathfrak{A}$ in the domain G such a bounded operator $\mathfrak{R}$ from $H_l(G, \Gamma)$ into $H_l(G)$ that $\mathfrak{A}\mathfrak{R} = I_1 + T_1$ and $\mathfrak{R}\mathfrak{A} = I_2 + T_2$, where I_1 and I_2 are the identity operators in $H_l(G, \Gamma)$ and $H_l(G)$, while T_1 and T_2 are bounded operators from these spaces respectively into $H_{l+1}(G, \Gamma)$ and $H_l(G)$.

Lemma 2. *The elliptic operator $\mathfrak{A}$ possesses a regularizer.*

For $p = 1$ the regularizer is constructed in (^{4}) and (^{21}). The latter construction is generalized here to the case $p \geqslant 1$ with the aid of Lemma 1.

First regularizers are constructed for (4) in R^n_+ and for $A_0(0, D)$ in R^n (their definitions are analogous to that just given). The first of them is constructed by the formulas

$$\mathfrak{R}(f, g) = \mathfrak{R}_0 f + \mathfrak{R}_1(g - B\mathfrak{R}_0 f); \tag{11}$$

$$\mathfrak{R}_0 f = MF^{-1} A_0^{-1}(0, \xi) |\xi|^{l_0} (1 + |\xi|^{l_0})^{-1} FLf; \tag{12}$$

$$\mathfrak{R}_1 g = F^{-1} \Omega(\xi, x^n) |\xi|^{l_0} (1 + |\xi|^{l_0})^{-1} Fg, \tag{13}$$

where F is the Fourier transform with respect to x; L is an operator extending functions from R^n_+ to R^n, bounded in the norm $\| \; \|_l$ (^{17}), and M is the operator restricting functions from R^n to R^n_+. The second is defined by formula (12), in which L and M are omitted. Next regularizers are constructed in R^n_+ and in

R^n for operators with coefficients close to constant ones. Finally, by means of a partition of unity, a regularizer for $\mathfrak{A}$ in G is defined.

Theorem 2. *Let* $\mathfrak{A}_1 = (A, B_1)$ *and* $\mathfrak{A}_2 = (A, B_2)$ *be elliptic operators with the same elliptic system* A *in* $\bar{G}$ *and different boundary operators* B_1 *and* B_2 *on* Γ. *Then*

$$\varkappa(\mathfrak{A}_1) - \varkappa(\mathfrak{A}_2) = \varkappa(S).$$

Here S *is an operator in the direct product of* r *spaces* $L^2(\Gamma)$, *defined by a square matrix of order* r *of singular integral operators. The symbol* σ_S *of this matrix is constructed explicitly from the matrices* A, B_1, *and* B_2 *on* Γ.

* The orders of rows in B_1 and B_2 with identical numbers may be different.

This symbol will be a nondegenerate square matrix of order r, defined and infinitely smooth on the bundle $\Xi(\Gamma)$ of tangent vectors to Γ of length 1. To define σ_s, let us return to the notation used in formulas (4)–(8). Let $B_{1,0}$ and $B_{2,0}$ be the principal parts of the operators B_1 and B_2; $\dot{\xi} = (\xi_1, \dots, \xi_{n-1})$ is the vector with components $(\xi_1, \dots, \xi_{n-1}, 0)$ in the coordinate system x, having length 1. Then (see (8))

$$\sigma_s(\dot{\xi}) = B_{1,0}\omega(\dot{\xi}) \cdot [B_{2,0}\omega(\dot{\xi})]^{-1}. \tag{14}$$

This definition does not depend on the choice of a stable basis.

Theorem 2 clarifies the dependence of the index of an elliptic operator on the boundary condition. In the case $p = 1$, a similar result was obtained in ([1]). Earlier, in ([9]), explicit formulas were found for the index of an elliptic operator in the two-dimensional case ($n = 2$) when $\max m_j < s$. We also note that in ([18]) the dependence of the index of the elliptic operator (A, B) on the coefficients of the system A inside the domain G is clarified.

For the proof of Theorem 2, we equalize the orders of the rows of the matrices B_1 and B_2 by multiplying them on the left the required number of times by the integro-differential operator Λ on Γ—the square root of $-\Delta$, where Δ is the Beltrami-Laplace operator (see ([6],[8])). After this we consider a linear deformation of B_1 into SB_2, applying Theorem 1.

Corollary. *If the symbol (14), considered as a continuous mapping of the bundle* $\Xi(\Gamma)$ *into the space of nondegenerate matrices of order* r, *is homotopic to a symbol depending only on the point of* Γ *(i.e., constant on tangent vectors issuing from one point on* Γ*), then* $\chi(\mathfrak{A}_1) = \chi(\mathfrak{A}_2)$.

It follows from this that the problem of homotopic classification of elliptic operators (A, B) with fixed A (this is one of the variants of the general problem posed in ([10])) is naturally solved by allowing integro-differential operators as B (cf. ([1],[6])).

Theorem 2 and the corollary from it allow one, for comparison of the indices of elliptic operators, to make use of the results of (^{19},^{20}); in particular, one obtains

Theorem 3. *The indices of the elliptic operators* (A, B_1) *and* (A, B_2) *coincide if 1)* $ps < n-1$ *or 2)* Γ *is homeomorphic to the* $(n-1)$*-dimensional sphere and* $ps < 2(n-1)$.

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