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Abstract

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ON AN ESTIMATE NEAR THE BOUNDARY OF A DOMAIN FOR A POLYHARMONIC FUNCTION AND ITS DERIVATIVES GIVEN ON A DISK

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In the literature (^{1}) there are estimates for the growth of solutions of elliptic equations of the second order near the boundary of the domain on which they are defined, obtained in the C metric on the basis, chiefly, of the study of the corresponding Green' s function. S. M. Nikol' skii (^{2}), by another method, obtained estimates, sharp in order, for the growth of a harmonic function and its derivatives near the boundary of a domain in the sense of L_p. The purpose of the present paper is to obtain analogous estimates in the L_2 metric for a polyharmonic function of arbitrary order on a disk (two-dimensional ball).

§ 1. Let u be an l-harmonic function in a disk of radius R, i.e., let it satisfy the equation:

$$\Delta^l u = 0; \tag{1,1}$$

$$I = \int_0^R \int_0^{2\pi} u^2 \rho \, d\rho \, d\theta < +\infty; \tag{1,2}$$

$$I_\rho^{(0)} = I_\rho = \int_0^{2\pi} u^2 \rho \, d\theta; \tag{1,3}$$

$$I_\rho^{(q)} = \int_0^{2\pi} u(q)^2 \rho \, d\theta, \tag{1,4}$$

where $q > 0$ is an integer; $u^{(q)}$ is any mixed partial derivative of the function u of order q.

In the present paper the inequality

$$I_\rho^{(q)} \leqslant \frac{C_{l,q}}{(R-\rho)^{2q+1}} I, \tag{1,5}$$

will be proved; it is sharp in order with respect to $R - \rho$ $(0 < \rho < R)$; $C_{l,q}$ is a constant depending only on l and q and not depending on R.

§ 2. We shall prove inequality (1,5) for $R = 1$. Then, by the change of variable $\rho = Rr$, we shall obtain (1,5) for arbitrary $R > 0$. Let $u(\rho, \theta)$ be an even l-harmonic function on the unit disk $(0 \leq \rho \leq 1)$. The assumption of evenness does not affect the generality of the argument. Then, as is known, u will have the form

$$u = \sum_{n=0}^{\infty} \left[a_{1n} + a_{2n}\rho^2 + \dots + a_{ln}\rho^{2(l-1)}\right] \rho^n \cos n\theta, \tag{2,1}$$

where a_{in} $(i = 1, 2, \dots, l)$ are linear combinations of the Fourier coefficients of the boundary functions, and

$$I = \int_0^1 \int_0^{2\pi} u^2 \rho \, d\rho \, d\theta = \pi \sum_{n=0}^{\infty} \Phi_n, \tag{2,2}$$

where

$$\Phi_n = 2 \int_0^1 [a_{1n} + a_{2n}\rho^2 + \dots + a_{ln}\rho^{2(l-1)}]^2 \rho^{2n+1} \, d\rho. \tag{2,3}$$

In what follows we shall compare terms of the series for one and the same n, and therefore, for brevity, instead of a_{in} we shall write a_i $(i = 1, 2, \dots, l)$. We have:

$$\Phi_n = \frac{A_1^2}{n+1} + \sum_{k=2}^{l} \frac{A_k^2 [(k-1)!]^2}{(n+k)^2 (n+k+1)^2 \dots (n+2k-2)^2 (n+2k-1)}, \tag{2,3'}$$

where

$$A_k = \sum_{i=k}^{l} \frac{(i-1)!}{(i-k)!(k-1)!} a_i \frac{(n+k) \dots (n+2k-1)}{(n+i) \dots (n+i+k-1)} \quad (1 \leq k \leq l). \tag{2,4}$$

Consider also

$$I_\rho = \int_0^{2\pi} u^2 \rho \, d\theta = \pi \sum_{n=0}^{\infty} F_n \rho^{2n+1} \quad (0 \leq \rho < 1), \tag{2,5}$$

where

$$F_n = [a_1 + a_2\rho^2 + \dots + a_l\rho^{2(l-1)}]^2. \tag{2,6}$$

The inequality holds

$$F_n \leq \sum_{k=1}^{l} 2^k (1-\rho^2)^{2k-2} B_k^2, \tag{2,7}$$

where

$$B_k \leq \sum_{i=k}^{l} a_i \, \frac{(i-1)!}{(k-1)!(i-k)!}. \tag{2,8}$$

Further:

$$B_k^2 \leq 2A_k^2 + \sum_{i=k}^{l-k} B_{k+i}^2 \, 2^{1+i} \frac{i^4 (i+1)^4 \ldots (i+k-1)^4}{(i!)^2 (n+2k)^2 (n+2k+1)^2 \ldots (n+2k+i-1)^2}, \tag{2,9}$$

where the A_k are defined by formula (2,4).

From (2,9) it follows that

$$B_k^2 \leq 2A_k^2 + \sum_{i=k+1}^{l} \frac{A_i^2}{(n+2k)^2 \ldots [n+2k+(i-k-1)]^2} \, \alpha_{k,i}, \tag{2,10}$$

where all $\alpha_{k,i}$ are positive integers independent of the number n.

Consider the auxiliary function

$$\Psi_k(\rho) = (1-\rho)^k \rho^{2n+1} \quad (0 \leq \rho \leq 1); \tag{2,11}$$

k, n are integers, $n \geq 0$, $k \geq 1$.

The inequality holds:

$$\Psi_k(\rho)(1+\rho)^{k-1} < \frac{L_k}{(n+1)^k}, \tag{2,12}$$

where

$$L_k = \frac{2^{k-1} k^k (1+k)^k}{e^k}. \tag{2,13}$$

Further, starting from (2.5) and taking into account (2.7), (2.10), and (2.12), we have

$$F_n \rho^{2n+1}(1-\rho) \leqslant \sum_{k=1}^{l} \frac{A_k^2 \beta_k}{(n+1)^{2k-1}}, \tag{2.14}$$

where $\beta_k > 0$ are bounded numbers independent of n, of the form

$$\beta_k = 2^k L_{2k-1} \alpha_{k,k} + 2^{k-1} L_{2k-3} \alpha_{k-1,k} + \cdots + 2L_1 \alpha_{1,k} \tag{2.15}$$

$$(k = 1, 2, \dots, l).$$

From (2.14) and (2.3) we find:

$$C_l = C_{l,0} = \beta_l \frac{(2l-1)[(2l-2)!]^2}{[(l-1)!]^4}. \tag{2.16}$$

Thus, inequality (1.5) for $q = 0$ and $R = 1$ is proved.

§ 3. For $R = 1$

$$I_\rho^{(q)} = \int_0^{2\pi} \left(\frac{\partial^q u}{\partial \rho^q} \right)^2 \rho \, d\theta = \pi \sum_{n=q}^{\infty} W_{qn} \rho^{2(n-q)+1} n^2 (n-1)^2 \cdots (n-q+1)^2, \tag{3.1}$$

where (for brevity we shall write simply W_q)

$$W_q = \left[\sum_{k=1}^{l} a_k \rho^{2k-2} \frac{(n+2k-2) \cdots (n+2k-q-1)}{n(n-1) \cdots (n-q+1)} \right]^2. \tag{3.2}$$

The inequality holds

$$D_k^2 \leqslant \sum_{i=k}^{l} B_i^2 \left(C_{i-1}^{k-1} \right)^2 \cdot 2^{i-k+1} (1-\rho^2)^{2(i-k)}, \tag{3.3}$$

where B_i are defined by formula (2.8), and

$$D_k = \sum_{i=k}^{l} a_i \rho^{2i-2k} C_{i-1}^{k-1}. \tag{3.4}$$

Further, the inequality holds

$$W_q \leqslant \sum_{i=0}^{q} \frac{\gamma_{i+1}}{(n+1)^{2i}} D_{i+1}^2, \tag{3.5}$$

where γ_{i+1} are positive constants independent of n.

Further, by virtue of (3.5), (3.3), and (2.12),

$$W_q(1-\rho)^{2q+1}n^2(n-1)^2\cdots(n-q+1)^2 \leqslant \sum_{k=1}^{l}\frac{A_k^2\delta_{k,q}}{(n+1)^{2k-1}}, \tag{3.6}$$

where the constants $\delta_{k,q} > 0$ do not depend on n.

Solving the inequality

$$\sum_{k=1}^{l} A_k^2\frac{\delta_{k,q}}{(n+1)^{2k-1}} \leqslant C_{l,q}\Phi_n, \tag{3.7}$$

where Φ_n are defined by formula (2.3), we find that

$$C_{l,q} = \delta_{l,q}\,\frac{[(2l-2)!]^2}{[(l-1)!]^4}. \tag{3.8}$$

Thus, inequality (1.5) is proved for $u^{(q)} = \partial^q u/\partial\rho^q$. For $u^{(q)} = \partial^q u/\partial\theta^i\partial\rho^{q-i}$, $(i = 1, 2, \dots, q)$, inequality (1.5) is obtained by the same method. By the change of variable $R\rho = r$ we obtain inequality (1.5) in the case of arbitrary R.

§ 4. From formula (3.8) it follows that

$$\lim_{l\to\infty} C_{l,q} = +\infty. \tag{4.1}$$

But in the course of proving relation (1.5), roughened inequalities may have been introduced. However, one can effectively indicate a sequence $l = l(N)$ of harmonic functions $(N = 1, 2, \dots)$, where $l(N) \to \infty$ as $N \to \infty$, and such a positive constant A, independent of N, that for

$$\rho = 1 - \frac{A}{N+1}$$

the inequality

$$\frac{l_\rho^{(0)}(1-\rho)}{I} > N, \tag{4.2}$$

holds, whence, obviously, (4.1) follows.

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CITED LITERATURE

[1] K. Miranda, *Partial Differential Equations of Elliptic Type*, IL, 1957. [2] S. M. Nikol' skii, *Siberian Mathematical Journal*, 1, No. 1 (1960).

Note: Figure translations are in progress. See original paper for figures.

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