



Soviet-era science, translated into English

MATHEMATICS

1962

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196201.16451>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

A. ZARELUA

EQUALITY OF DIMENSIONS AND BICOMPACT EXTENSIONS

(Presented by Academician P. S. Aleksandrov on 11 I 1962)

M. Katětov proved * that if a bicomactum X admits a zero-dimensional mapping ** into the Hilbert cube, then $\text{ind } X = \dim X = \text{zw } X$ ***, where $\text{zw } X$ is the least of all those cardinal numbers τ such that the bicomactum X admits a zero-dimensional mapping into the Tikhonov cube I^τ . Here this result is extended to a considerably wider class of spaces ****: to fully paracompact ***** spaces admitting decomposing mappings (see below) into the Hilbert cube, in particular to fully paracompact subspaces of locally bicomact finite-dimensional groups. At the end several results are given on the extension of mappings (in particular, decomposing ones) to bicomact extensions. The theorems of S. Mardešić, J. Nagata, and E. Sklyarenko ***** are strengthened.

A. A mapping f of a space X into a space Y is called **decomposing** if for every point $x \in X$ and every neighborhood Ox of it there exists such a neighborhood Oy of the point $y = fx \in Y$ that its full preimage decomposes into the sum of two disjoint open sets G and H in such a way that $x \in G \subset Ox$.

It is easy to prove the following propositions:

A1. *Every closed zero-dimensional mapping of a normal space is a decomposing mapping.*

A2. *Every decomposing mapping is zero-dimensional *****.*

A3. *A mapping of a locally bicomact space is decomposing if and only if it is zero-dimensional *****.*

A4. *If a decomposing mapping of a space X is considered on a subset $M \subset X$, then the resulting (restricted) mapping is also decomposing.*

A5. *If the space X has an open covering $\{\Gamma_\alpha\}$ and a family of mappings f_α such that each f_α is decomposing on the corresponding element Γ_α , then the product $f = \{f_\alpha\}$ of these mappings is also decomposing.*

Using proposition A4, one easily proves by induction:

Theorem 1. *If there exists a decomposing mapping of the space X into a space Y , then $\text{ind } X \leq \text{ind } Y$. Let $\text{zw } X$ be the least of all such cardinal numbers τ*

that the space X admits a decomposing mapping into the Tikhonov cube I^τ (the product of τ intervals).

Corollary. Always $\text{ind } X \leq \text{zw } X$.

* See ⁽¹⁾, Theorems 3 and 4, and also ⁽²⁾, the remark to Theorem 4.

** All mappings and functions occurring here are assumed continuous. A mapping is zero-dimensional if the full preimages of all points have small inductive dimension (Urysohn-Menger) $\text{ind } f^{-1}y = 0$.

*** By dim is denoted the dimension defined by means of the multiplicity of finite open coverings.

**** By a space here and throughout is meant a Hausdorff topological space.

***** See ⁽³⁾, § 2, or here item B.

***** See ⁽⁴⁾, Theorem 2; ⁽⁵⁾, Theorem 2; ⁽⁶⁾.

***** As will be shown below, decomposing mappings are those mappings which can be extended to zero-dimensional mappings on bicomact extensions of the given spaces.

***** Proposition A3 belongs to Yu. M. Smirnov.

For brevity, let us call a space X admitting a disjoint mapping into Hilbert cube (i.e., $\text{zw } X \leq \aleph_0$) a Z -space*.

Theorem 2. For every normal Z -space X one always has

$$\text{zw } X \leq \text{dim } X.$$

Corollary. For every normal Z -space one always has

$$\text{ind } X \leq \text{dim } X.$$

The proof of the theorem is based on a theorem of M. Katětov on γ -zero-dimensional mappings of an n -dimensional space into E^{n**} ($0 \leq n \leq \infty$).

B. We shall say that a cover β is **weakly inscribed** in a cover α if β contains a subcover inscribed in α . We shall call a space **fully paracompact***** if it is regular and every open cover of it admits a weakly inscribed open cover that decomposes into the sum of a countable number of star-finite**** subcovers*****.

The following propositions are true:

B1. If X is fully paracompact, then $\text{dim } X \leq \text{ind } X$.

B2. If in a fully paracompact space X the sum theorem for a countable number of closed sets holds for the small inductive dimension, then

$$\text{ind } X = \text{Ind } X*****.$$

Hence we obtain the first main result:

Theorem 3. *If X is a fully paracompact Z -space, then*

$$\text{ind } X = \text{zw } X = \text{dim } X = \text{Ind } X. \quad (1)$$

Corollary. *For every metrizable fully paracompact space the equalities (1) hold*****.*

B3. *A normal space possessing a countable open cover $\{\Gamma_i\}$, in which the closure of each element Γ_i is a Z -space, is itself also a Z -space.*

The proof is carried out with the aid of proposition A5.

Theorem 4. *A normal space possessing a locally finite cover, each element of which is a Z -space, is itself a Z -space.*

Corollary 1. *Every normal homogeneous space of any topological group that possesses a neighborhood of the identity which is a Z -space is itself also a Z -space.*

Since every finite-dimensional locally bicomact group possesses a neighborhood of the identity which is a Z -space, we have:

Corollary 2. *Every homogeneous space of any finite-dimensional locally bicomact group is a Z -space*****.*

Corollary 3. *For every fully paracompact (in particular, for every closed or open paracompact) subset X of*

* Every Z -space is completely regular, since under disjoint mappings complete regularity (and some other separation axioms) is transferred from the image fX to the preimage X .

** See (7), pp. 353 and 357. E^n is Euclidean n -dimensional space, E^∞ is Hilbert space.

*** See (3), § 2.

**** A cover is star-finite if each of its elements intersects only finitely many other elements of the same cover.

***** Every strongly paracompact space, in particular every regular finally compact and every bicomact space, is fully paracompact. There exist metrizable fully paracompact spaces not representable as a sum of a countable number of closed strongly paracompact subspaces (see (3), § 2).

***** See (3), Corollary 2 of Theorem 6, and also Lemma 7, from which proposition B2 is obtained by a simple induction.

***** By a theorem of M. Katětov (see (7), Theorem 2.15), every metrizable space can be zero-dimensionally mapped into Hilbert space. It is easy to see that every uniformly zero-dimensional mapping is disjoint.

***** While this work was being prepared, E. Sklyarenko proved that every such homogeneous space is the skew product of some Euclidean space with some generalized Cantor set D^τ (of dimension 0), whence Corollary 2 follows directly. of an infinite-dimensional locally bicomact group (or any of its homogeneous spaces) the equalities (1)* hold.

The main case of the theorem consists in the fact that B is an extension of the image $Y = fX$.

Theorem 5 (Urysohn's inequality). For any finite covering $\{M_i\}$ of a normal space X (by arbitrary sets) one always has

$$\dim X \leq \sum_{i \leq k} \dim M_i + k + 1,$$

where k is the number of the set M_i .

Theorem 6. For every locally bicomact paracompact** Z -space X , its dimension is equal to the least of all such natural numbers k that there exist $k + 1$ (paracompact) sets M_i such that $\dim M_i = 0$, where $1 \leq i \leq k + 1$, and

$$X = \bigcup_{i \leq k+1} M_i.$$

B. The proofs of the following theorems on extensions require the apparatus of the theory of rings of functions.

Theorem 7. Let there be given a decomposing mapping f of the space X into the bicomact B ; then there exist a bicomact C , bicomact extensions bX and aX , and also mappings \tilde{f}, π, g and \mathfrak{D} , such that the diagram

$$\begin{array}{ccc} & \tilde{f} & \\ bX & \longrightarrow & B \\ \pi \downarrow & & \downarrow \mathfrak{D} \\ aX & \xrightarrow{g} & C \end{array}$$

has the following properties: 1) π is the natural*** projection of the extension bX onto the extension aX ; 2) \tilde{f} is an extension of the mapping f ; 3) the mappings \tilde{f} and g are zero-dimensional; 4) the diagram is commutative; 5) $\dim C = \dim B$; 6) the weight of the extension bX is equal to the maximum of the weights of the spaces X and B ; 7) the weights of the spaces X and aX are equal.

The main case of the theorem consists in the fact that B is an extension of the image $Y = fX$.

Corollary 1. A mapping of a space X into a completely regular space Y is decomposing if and only if it can be extended to a zero-dimensional mapping onto some bicomact extension.

Corollary 2. A space X is a Z -space if and only if it has a bicomact extension that is a Z -space.

Corollary 3. Every space X has such a bicomact extension bX of the same weight that $zw bX = zw X$.

Corollary 4. Every metric space X has a bicomact extension that is a Z -space, of the same weight and the same dimension as X .

Corollary 5. Let f be a decomposing mapping of a space X onto a space Y , and let B be a bicomact extension of the space Y ; then there exists such a bicomact extension bX of the space X , to which the mapping f is extendable to a zero-dimensional mapping, and such that the weight of the extension bX is equal to the maximum of the weights of the spaces X and B .

Corollary 6. Let f be a decomposing mapping of the space X onto the space Y ; then there exists a bicomact extension aX of the space X of the same weight as X , and such that $\dim aX \leq \dim Y$.

Theorem 8. Let f be a mapping of a normal space X into a bicomact B , and let $\{F_\lambda\}$ be a system of closed sets of the space X ,

* In particular, we obtain also the known theorem of B. Pasynkov (see ⁽⁸⁾) on the equality of dimensions for finite-dimensional locally bicomact groups.

** By the well-known theorem of Morita, the space X will be strongly paracomact, and hence also fully paracomact.

*** That is, leaving the points of the space X fixed.

whose cardinality is no greater than the weight of the compactum B ; then there exist a compactification aX of the space X , a compactum C , and mappings \tilde{f} , g , and h such that the diagram

$$\begin{array}{ccc} & \tilde{f} & \\ bX & \longrightarrow & B \\ g \searrow & & \nearrow h \\ & C & \end{array}$$

has the following properties: 1) \tilde{f} is an extension of the mapping f ; 2) the diagram is commutative; 3) $\dim bX[F_\lambda] = \dim F_\lambda = \dim C[gF_\lambda]$ for each λ ; 4)

the weight of the extension bX is equal to the maximum of the weights of the spaces X and B ; 5) the weights of the compacta B and C are equal.

The basic case of the theorem consists here as well in the fact that one may take the expansion of the image $Y = fX$. The space X may be regarded as an element of the system $\{F_\lambda\}$ —then we obtain one more condition *: $\dim bX = \dim X = \dim C$.

Corollary **. Let bX be a compactification of a normal space X , and let $\{F_\lambda\}$ be a system of closed subsets of the space X whose cardinality is no greater than the weight of the extension bX ; then there exists a compactification cX of the space X , following bX , such that $\dim cX[F_\lambda] = \dim F_\lambda$ and its weight is equal to the weight of the extension bX .

Similarly the following is proved.

Theorem 9. Let bX be a compactification of a normal space X , and let $\{F_\lambda\}$ be a system of closed subsets of the space X whose cardinality is no greater than the weight of the space X ; then there exists a compactification aX of the space X , preceding the extension bX , such that $\dim aX[F_\lambda] = \dim bX[F_\lambda]$ and its weight is equal to the weight of the space X .

The following theorem strengthens one result of Nagata ***.

Theorem 10. For every metric space X one always has $\text{ind } X \leq \text{ad}(U(X), C(X))$ ****.

Corollary. If for a metric space X the equality $\text{ind } X = \dim X$ holds (in particular, if X is completely paracompact), then $\text{ind } X = \text{ad}(U(X), C(X))$.

Moscow State University
named after M. V. Lomonosov

Received
22 XII 1961

CITED LITERATURE

1. M. Katětov, Časop. pěst. mat. fys., **75**, 1 (1950).
2. M. Katětov, Časop. pěst. mat. fys., **75**, 79 (1950).
3. A. Zarelua, DAN, **141**, No. 4, 15 (1961).
4. Š. Mardešić, Illinois J. Math., **4**, No. 2, 278 (1960).
5. Ju. Nagata, Proc. Japan Acad., **36**, No. 2, 49 (1960).
6. E. Sklyarenko, DAN, **123**, No. 1, 36 (1958).

7. M. Katetov, Czechoslovak Math. J., **2** (77), 333 (1952).

8. B. Pasynkov, DAN, **132**, No. 5, 1035 (1960).

* Therefore this theorem is a direct strengthening of the well-known theorem of C. Mardešić; see (4), Theorem 2.

** This corollary strengthens the well-known theorem of E. Sklyarenko; see (6), Theorem 2.

*** See (5).

**** Here $C(X)$ denotes the ring of all bounded (continuous and real-valued) functions on X , and $U(X)$ the subset of this ring consisting of all uniformly continuous functions. For the definition of the dimension $\text{ad}(U(X), C(X))$, see Nagata's paper (5), where he proves the assertion of the corollary for locally compact metric spaces.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.