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## Abstract

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# LOCAL PROPERTIES OF CERTAIN APPROXIMATION OPERATORS

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**1.** The question of the local properties of certain approximation operators has its origin in Riemann' s theorem on the local convergence of a Fourier series. In the present note we refine the formulation of the question: we are interested in the extent to which the rate of convergence of a given sequence of approximating functions at some point $x_0$ is determined by the properties of the function being approximated in some neighborhood of the point $x_0$.

Let $Q_n(f)$ be a sequence of homogeneous operators, defined on some class $\mathfrak{M}$ of functions $f$ whose domain of definition is the interval $[a, b]$ $(-\infty < a < b < +\infty)$.

**Definition.** The local characteristic of the operators $Q_n$ on the class $\mathfrak{M}$ will mean the function

$$\varphi(n, \delta) = \sup_{x_0 \in [a,b]} \sup_f |Q_n(f; x_0)|,$$

where the first supremum is taken over all such functions $f$ from $\mathfrak{M}$ that $f(x) = 0$ for points $x$ from $[a, b]$ lying on the interval $[x_0 - \delta, x_0 + \delta]$. In other words, $\varphi(n, \delta)$ gives an upper estimate for the approximation at the point $x_0$ of any function from $\mathfrak{M}$ that is equal to zero on the interval $|x - x_0| \leq \delta$, if nothing is known about the properties of the function outside this interval.

Denote by $S(B)$ the unit sphere of the normed space $B$. Below, as $\mathfrak{M}$, we shall take the unit spheres of the following spaces: the spaces $C^k[a, b]$ of functions having $k$ continuous derivatives, where $S(C^k[a, b])$ is defined by the conditions $\max |f^{(i)}(x)| \leq 1$ $(a \leq x \leq b;\ 0 \leq i \leq k)$; the spaces $C^k_\omega[a, b]$ of functions from $C^k[a, b]$ for which $\omega(f^{(k)}; \varepsilon) \leq M\omega(\varepsilon)$, and $S(C^k_\omega[a, b])$ is the set $\{f : \|f\|_{C^k[a,b]} \leq 1;\ \omega(f^{(k)}; \varepsilon) \leq \omega(\varepsilon)\}$; the spaces $M[a, b]$ of measurable bounded functions, where $S(M[a, b])$ is the set $\{f : \max |f(x)| \leq 1\ (a \leq x \leq b)\}$; and the spaces $L(a, b)$ of integrable functions with the usual metric. If in the preceding definitions $a = -2\pi$, $b = 2\pi$, and the functions $f$ are $2\pi$-periodic, then the corresponding spaces will be denoted by $\widetilde{C}^k$, $\widetilde{C}^k_\omega$, $\widetilde{M}$, $\widetilde{L}$.

The significance of the definition just given is shown by the following theorem.

**Theorem 1.** Let $B$ be a normed space of functions defined on $[a,b]$, and suppose that $S(C^k[a,b]) \subset S(B)$; let the sequence of operators $Q_n(f)$, defined on $B$, satisfy the condition

$$|Q_n(f_1+f_2)-(f_1+f_2)| \le |Q_n(f_1)-f_1| + |Q_n(f_2)-f_2|, \tag{1}$$

where $f_1$ and $f_2$ are from $B$, and let $\varphi(n,\delta)$ be the local characteristic of $Q_n$ on the class $S(B)$. Then, if the function $f$ from $B$ on the interval $\Delta = [x_0-\delta, x_0+\delta]$ belongs to $C^k_\omega[\Delta]$, then at the point $x_0$ the following estimate holds

$$|f(x_0) - Q_n(f;x_0)| \le$$

$$\le N_1(k)\left[\|f\|_{C^k_\omega[\Delta]}\, \mathcal{E}\{C^k_\omega[a,b];Q_n\} + \left(\|f\|_{C^k_\omega[\Delta]} + \|f\|_B\right)\varphi(n,\delta)\right],$$

where

$$\mathcal{E}(B;Q_n) = \sup_{f\in S(B)}\sup_{x\in[a,b]}\|f(x)-Q_n(f;x)\|.$$

We see that the estimate of the magnitude of approximation of the function $f$ by the expressions $Q_n(f)$ at the point $x_0$ consists of two terms, of which one depends only on the properties of the function on the interval $|x-x_0| \le \delta$, while the other is determined by the magnitude of the local characteristic.

**Remark.** Condition (1) is obviously satisfied for additive operators $Q_n$; moreover, it is also satisfied if $Q_n(f)$ is a polynomial of best approximation for the function $f$.

**2.** We give examples of operators for which the local characteristic decreases as $n \to \infty$ faster than a certain geometric progression.

**Theorem 2.** For the Bernstein polynomials

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

the local characteristic on the sphere $S(M[0,1])$ admits the estimate $\varphi(n,\delta) \le en^2[q(\delta)]^n$, where

$$q(\delta) = \max_{0\le t,\ x\le 1;\ |x-t|\le\delta} x^t t^{-t}(1-x)^{1-t}(1-t)^{t-1} < 1.$$

This theorem follows easily from the estimates given in (1).

**Theorem 3.** For the Vallée-Poussin polynomials

$$V_n(f;x) = \frac{(2n-1)!!}{(2n)!!} \int_{-\pi}^{\pi} \cos^n t\, f(x+t)\, dt$$

the local characteristic on the sphere $S(\widetilde{M})$ admits the estimate $\varphi(n,\delta) \leq N_2\sqrt{n}\cos^n\delta$.

Let us note that in both examples the operators approximate functions very slowly; for example, the Bernstein polynomials for any function $f(x) \neq ax+b$ give an order of approximation no higher than $1/n$.

**3.** We estimate the local characteristic of operators that give, on some classes of functions, approximation of the order of best approximation.

**Theorem 4.** For the operators $T_{n(r,k)}(f)$, defined by the equality

$$T_{n(r,k)}(f;x) = \int_{-\pi}^{\pi} K_n(t) \sum_{\nu=1}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x+\nu t)\, dt,$$

where $K_n(2t) = C_{n,r}(\sin nt/\sin t)^{2r}$, and $C_{n,r}$ is found from the condition $\|K_n\|_L \leq 1$, the local characteristic on the sphere $S(\widetilde{L})$ admits the estimate $\varphi(n,\delta) \leq N_3(r)k^{2r}n^{-(2r-1)}\delta^{-2r}$, while the local characteristic on the sphere $S(\widetilde{C}^l_\omega)$ admits the estimate $\varphi(n,\delta) \leq N_4(r)k^{2r}n^{-2(r-1)}\omega(n^{-1})\delta^{-2r}$.

Let us note that $T_{n(r,k)}$ are trigonometric polynomials of order $r(n-1)$ and give approximation of the order of best approximation on the classes $\widetilde{C}^k_\omega$, if $k \leq 2r-2$.

**4.** In this section we consider an operator that is remarkable in certain respects. In doing so we use an idea of N. I. Akhiezer, applied in the paper (2).

Let $\psi(t)$ be an even infinitely differentiable function, with $\psi(t) = 0$ for $|t| \geq 1$; $\psi(0) = 1/\sqrt{2\pi}$; $\psi^{(n)}(0) = 0$ $(n = 1, 2, \ldots)$; let

$K(u)$ is the Fourier transform of the function $\psi(t)$. We define the operator $\Phi_n$ in the space $M(-\infty,\infty)$ by the equality

$$\Phi_n(t;x) = \int_{-\infty}^{\infty} f\left(x + \frac{t}{n}\right) K(t)\, dt = n \int_{-\infty}^{\infty} f(t) K(n(x-t))\, dt.$$

It is not difficult to obtain the following properties of the function $K(u)$:

1) $K(u)$ is an even entire function of exponential type of degree $\leqslant 1$.

2) $|K(u)| \leqslant N_5(p)(1+|u|)^{-p}$ for any $p \geqslant 0$.

3)
$$\int_{-\infty}^{\infty} K(u)\,du = 1; \qquad \int_{-\infty}^{\infty} u^m K(u)\,du = 0 \quad \text{for integers } m > 1.$$

Relying on these, one can establish the following. For any function $f$ in $M[-\infty, \infty]$, $\Phi_n(f)$ is an entire function of exponential type of degree $\leqslant n$. If $f$ is even, then $\Phi_n(f)$ is also even; if $f$ is $2\pi$-periodic, then so is $\Phi_n(f)$, i.e., for $f \in \widetilde{M}$, $\Phi_n(f)$, by S. N. Bernstein' s theorem ([3]), is a trigonometric polynomial of order $\leqslant n$. The uniform approximation by the polynomials $\Phi_n(f)$ of functions from $\widetilde{C}^k_\omega$ has the order of best approximation for arbitrary $k$ and $\omega$, i.e., for $f \in \widetilde{C}^k_\omega$ we have:

$$\|f - \Phi_n(f)\|_{\widetilde{C}} \leqslant N_6(k)\,\omega(1/n)\,n^{-k}\,\|f\|_{\widetilde{C}^k_\omega},$$

where $N_6(k)$ does not depend on $n$ and $u$. The local properties of the operators $\Phi_n(f)$ are characterized by Theorem 5.

**Theorem 5.** *For the polynomials $\Phi_n(f)$ on the sphere $S(\widetilde{L})$, the local characteristic admits the estimate $\varphi(n, \delta) \leqslant N_7(p)(n\delta)^{-p}$, and on the sphere $S(\widetilde{C}^k_\omega)$ the estimate $\varphi(n, \delta) \leqslant N_8(p)(n\delta)^{-p} n^{-k}\omega(1/n)$, where p is any positive number.*

Comparing the first of these estimates with Theorem 1, we obtain the following result.

**Theorem 6.** *Let $f \in \widetilde{L}$, and suppose that on some interval $[c, d]$ we have $f(x) \in C^k_\omega[c, d]$. Then at any point $x$ lying inside the interval $[c, d]$, the estimate holds:*

$$|f(x) - \Phi_n(f; x)| = O\{n^{-k}\omega(1/n)\}. \tag{2}$$

Roughly speaking, the approximating sequence $\Phi_n(t)$ is such that the order of approximation at each point is determined by the properties of the function in an arbitrarily small neighborhood of that point. This thereby refines the result from ([4]), where likewise for a function $f$ ($f \in \widetilde{L},\ f \in C^k_\omega[c, d]$) the existence was asserted of a sequence of trigonometric polynomials which, inside $[c, d]$, gives an approximation estimated by formula (2); however, this sequence depended, generally speaking, on the interval $[c, d]$ and on $k$.

5. Let $T_n(f)$ be polynomials of best approximation, and let $\mathfrak{M}$ be an arbitrary class of functions such that $\mathfrak{M} \subset S(\widetilde{C})$. Using the result of work ([5]), for the local characteristic of the polynomials $T_n(f)$ on the class $\mathfrak{M}$ we obtain, by virtue of Theorem 1, the estimate $\varphi(n, \delta) \geqslant \alpha\mathcal{E}\{\mathfrak{M}; T_n\}$ for some sequence $n = n_1, n_2, ...$, with $\alpha > 0$.

Roughly speaking, in approximating a continuous function by the polynomials $T_n(t)$, we cannot, by virtue of increased smoothness of the function in a neighborhood of some point, obtain at that point an approximation faster than the uniform approximation of the function.

6. The results set forth, especially item 4, find application in approximation theory; they make it possible to prove comparatively simply the basic-

results of the work [6], and also to obtain some constructive characteristics of functions of many variables differentiable in a closed domain. In [1], for studying approximation by positive operators, the quantity $\alpha_n(\delta)$ is considered, which is in fact a local characteristic of operators on the sphere $S(M)$.

Let us also note the communication [7] on the work [8], in which the author, applying a certain method of summing a Fourier series, constructs the operator $A_n(f)$ and proves for it an assertion somewhat more precise than Theorem 6 for $\Phi_n(f)$. From this result there also follows a certain estimate for the local characteristic of $A_n(f)$; however, the dependence $\varphi(n, \delta)$ on $\delta$, important in applications, is not estimated.

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