ON COMPARING THE COMPLEXITY OF REALIZING MONOTONE FUNCTIONS BY CONTACT CIRCUITS CONTAINING ONLY CLOSING CONTACTS AND BY ARBITRARY CONTACT CIRCUITS

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Abstract

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ON COMPARING THE COMPLEXITY OF REALIZING MONOTONE FUNCTIONS BY CONTACT CIRCUITS CONTAINING ONLY CLOSING CONTACTS AND BY ARBITRARY CONTACT CIRCUITS

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As is known, every monotone function ([1–3]) can be realized by a contact circuit (in the sense of ([3]), pp. 40–41) containing only closing contacts. Yu. L. Vasiliev constructed an example of a function for which the minimal realization using only closing contacts is more complicated than a minimal realization containing opening contacts (see Fig. 1). Therefore the question arises of comparing the complexities of such realizations. For an exact formulation of the problem we introduce the following notation. $L(S)$ is the number of contacts in the circuit S; $L^+(f)$ (respectively $L(f)$) is the minimal number of contacts sufficient for realizing the function f by a contact circuit containing only closing contacts (respectively by an arbitrary contact circuit); $\lambda(n) = \max L^+(f)/L(f)$ (the maximum is taken over all monotone functions f of n arguments). N. A. Karpova showed ([4]) that $\lambda(n) \geq 3/2$.

The purpose of the present work is to prove the assertion:

Theorem 1. $\lambda(n) \to \infty$ $(n \to \infty)$.

Theorem 1 shows that some monotone functions can be realized, using opening contacts, essentially more simply than by circuits containing only closing contacts.

The proof of the theorem is based on the construction of a sequence $f_1, f_2, \dots, f_n, \dots$ of functions (f_n depends on the arguments $x_1, \dots, x_n$) such that, on the one hand, $L(f_n) \leq C_1 n$, where C_1 is some constant, and, on the other hand, $L^+(f_n)/n \to \infty$.

Consider the function

$$F_n(x_1, \dots, x_n) = \bigvee_{1 \leq i < j \leq n} x_i x_j, \qquad n \geq 2.$$

It is obvious that $F_n(x_1, \dots, x_k, 0, \dots, 0) = F_k(x_1, \dots, x_k)$. Since $F_n(x_1, \dots, x_n)$ depends essentially on all its arguments, it follows that

$$L^+(F_n) \geq n. \tag{1}$$

Let S be an arbitrary contact circuit for F_n, containing only closing contacts, and let α be (one of) its poles. Consider the set $\mathfrak{M}$ of vertices of the circuit S, each of which is connected with α by an edge (or by a bundle of parallel edges) –see Fig. 2. Let the pole α and a vertex β from $\mathfrak{M}$ be connected by a bundle P_β of parallel contacts, and let Q_β be the set of the remaining contacts incident with β. If in the set Q_β there are contacts of the same variables as in P_β (Fig. 3a), then each such contact can be "thrown over" by one end from the vertex β to the pole α (Fig. 3b; cf. ([5])). In this operation the number of contacts and the conductivity of the circuit do not change. In what follows we shall consider circuits in which all such transformations have already been carried out; such circuits will be called **reduced**. The maximum number of contacts in a bundle P_β will be called the **rank** of the pole α.

Lemma 1. Let a circuit S realize the function F_n, contain only make contacts, and let one of its poles α have rank k. Then $L(S) \geq n^2/8k$.

Proof. Since S contains no break contacts, each of the C_n^2 conjunctions in the reduced disjunctive normal form of the function F_n (which, by the monotonicity of F_n, coincides with the minimal disjunctive normal form–see, for example, (6)) corresponds to a chain whose conductivity is equal to this conjunction. In the circuit S, choose one chain for each such conjunction. Let the bundle P_β contain k_β contacts, and let the set Q_β contain l_β contacts (β is an arbitrary vertex of $\mathfrak{M}$). It is clear that the number of selected chains passing through the bundle P_β does not exceed $k_\beta l_\beta$. And since there are C_n^2 selected chains in all,
$\sum_\beta k_\beta l_\beta \geq C_n^2$. Since $k_\beta \leq k$, it follows that
$\sum_\beta l_\beta \geq \frac{1}{k} C_n^2$. On the other hand, each contact from $\bigcup_\beta Q_\beta$ belongs to no more than two sets Q_β. Consequently, the number of all contacts in $\bigcup_\beta Q_\beta$ is not less than $\frac{1}{2}\sum_\beta l_\beta$, and

$$L(S) \geq \frac{1}{2}\sum_\beta l_\beta \geq \frac{n^2 - n}{4k} \geq \frac{n^2}{8k}.$$

The lemma is proved.

Fig. 1
Fig. 1

Fig. 2
Fig. 2

Fig. 3
Fig. 3

Consider the function

$$\varphi_{n,a}(k) = a\left((n-k)\frac{\ln(n-k)}{\ln\ln(n-k)} + k\frac{\ln k}{\ln\ln k}\right) + k.$$

Lemma 2. The function $\varphi_{n,a}(k)$, under

$$0 \le a \le 1, \tag{2}$$

$$n\ln\ln n/8a\ln n \le k \le n/2 \tag{3}$$

is nondecreasing, provided $n \ge N_1$ (where N_1 is a natural number independent of a).

Proof. It is enough to show that, under the indicated conditions, $\varphi'_{n,a}(k) \ge 0$. We have*

$$\begin{aligned}
\varphi'_{n,a}(k) &= a\left(\frac{\ln k}{\ln\ln k} - \frac{\ln(n-k)}{\ln\ln(n-k)}\right) \\
&\quad + a\left(\frac{1}{\ln\ln k}\left(1-\frac{1}{\ln\ln k}\right) - \frac{1}{\ln\ln(n-k)}\left(1-\frac{1}{\ln\ln(n-k)}\right)\right) + 1 \\
&\ge^{**} a\left(\frac{\ln k}{\ln\ln k} - \frac{\ln(n-k)}{\ln\ln(n-k)}\right) + 1 \\
&\ge^{***} a\left(\frac{\ln k}{\ln\ln k} - \frac{\ln n}{\ln\ln n}\right) + 1 \\
&\ge^{****} a\left(\frac{\ln n - \ln\ln n}{\ln\ln n} - \frac{\ln n}{\ln\ln n}\right) + 1 = 1 - a \ge 0.
\end{aligned}$$

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* $A \ge B$ means that, for sufficiently large n, the inequality $A \ge B$ holds.

** The function $\frac{1}{x}\left(1-\frac{1}{x}\right)$ does not increase for sufficiently large x; moreover, by (3), $k \le n-k$ and $k \to \infty$ as $n \to \infty$.

*** The function $\ln x/\ln\ln x$ does not decrease for sufficiently large x; moreover, $n/2 \le n-k \le n$.

**** By (3), (2),

$$\ln k \ge \ln n + \ln\ln\ln n - \ln\ln n - \ln 8a \ge \ln n + \ln\ln\ln n - \ln\ln n - \ln 8 \ge \ln n - \ln\ln n.$$

Moreover, $\ln\ln k \le \ln\ln n$.

The lemma is proved.

Theorem 2*.

$$L^+(F_n) \geq cn \ln n / \ln \ln n, \tag{4}$$

where c is a certain positive constant.

Proof. Let N be a natural number satisfying the conditions ** a) $N \geq N_1$ (see Lemma 2); b) $\ln \ln n \geq 8$ for $n \geq N$; c) the function $\ln \ln n / \ln n$ is decreasing for $n \geq N$; d) $\min_{3 \leq n \leq N} \ln \ln n / \ln n = \ln \ln N / \ln N$; e) $\ln \ln N / \ln N \leq 1/2$.

Put now

$$c = \ln \ln N / \ln N. \tag{5}$$

Then we have:

1)

$$0 < c \leq 1/2 \tag{6}$$

(see 5), 6), e)); 2) for $n > N$

$$2 \leq n \ln \ln n / 8c \ln n \leq n/8 < n/2 \tag{7}$$

(the first inequality, since $n > \ln n$, follows from b) and (6); the second—from c) and (5)); 3) for $3 \leq n \leq N$

$$L^+(F_n)/(n \ln n / \ln \ln n) \geq \ln \ln n / \ln n \geq c \tag{8}$$

(see (1), d), (5)).

We shall now prove inequality (4), with the constant c defined above, by induction on n.

I. For $n = 2$ inequality (4) is trivial. For $3 \leq n \leq N$, inequality (4) follows from (8).

II. Suppose now that (4) has been proved for $2, \dots, n-1$ $(n-1 \geq N)$. We shall prove it for n. Let S be some minimal reduced circuit for the function F_n, α one of its (two) poles, and k_0 the rank of this pole. Two cases are possible.

A. $k_0 \leq n \ln \ln n / 8c \ln n$. Then, by Lemma 1, $L^+(F_n) = L(S) \geq n^2/8k_0 \geq cn \ln n / \ln \ln n$, and inequality (4) is valid.

B.

$$k_0 > n \ln\ln n / 8c \ln n. \tag{9}$$

Take in the circuit S a bundle P of k_0 parallel contacts adjacent to the pole α. By the symmetry of the function F_n, we may assume that P contains the contacts $x_1, \dots, x_{k_0}$. Let $k = \min(k_0, [n/2])$. By (7), (9),

$$n \ln\ln n / 8c \ln n \le k \le n/2, \tag{10}$$

$$2 \le k < n, \qquad 2 \le n-k < n. \tag{11}$$

Substitute zeros in the circuit S in place of the variables $x_{k+1}, \dots, x_n$. Then from the circuit S we obtain a circuit S' for the function $F_k(x_1, \dots, x_k)$, containing

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* It can be shown that

$$L^+(F_n) \le C_2 n \ln n, \tag{*}$$

where C_2 is a certain constant. Indeed,

$$F_{2n}(x_1, \dots, x_{2n}) = (x_1 \vee \dots \vee x_n)(x_{n+1} \vee \dots \vee x_{2n}) \vee F_n(x_1, \dots, x_n) \vee$$

$$\vee F_n(x_{n+1}, \dots, x_{2n}).$$

Therefore $L^+(F_{2n}) \le 2n + 2L^+(F_n)$. Hence, by induction on m, it is proved that $L^+(F_{2^m}) \le m2^m$. From the last inequality (*) follows easily.

** Conditions a)—e) are dependent. The superfluous conditions have been added for greater simplicity of the subsequent exposition and to make the proof of Theorem 2 independent of the proof of Lemma 2.

branch* (7) of k contacts connected in parallel $x_1, \dots, x_k$. Therefore $L(S') \geqslant L^+(F_k) + k$. On the other hand, if zeros are substituted in the circuit S in place of $x_1, \dots, x_k$, one obtains a circuit for the function $F_{n-k}(x_{k+1}, \dots, x_n)$, containing only contacts of the variables $x_{k+1}, \dots, x_n$. Thus,

$$L^+(F_n) = L(S) \geqslant$$

$$\geqslant L^+(F_{n-k}) + L^+(F_k) + k.$$

Fig. 4

Figure 1: Fig. 4

From the last inequality, by virtue of the induction hypothesis and (11), we have $L^+(F_n) \geqslant \varphi_{n,c}(k)$. Therefore, by Lemma 2 (and taking account of (10) and (6)),

$$L^+(F_n) \geqslant \varphi_{n,c}(n \ln\ln n/8c\ln n). \tag{12}$$

Fig. 4

From (7) we have**

$$\ln(1 - \ln\ln n/8c\ln n) > -\ln\ln n/4c\ln n. \tag{13}$$

Moreover, taking (7) into account, we have

$$\ln\ln\left(n - \frac{n\ln\ln n}{8c\ln n}\right) \leqslant \ln\ln n, \qquad \ln\ln\frac{n\ln\ln n}{8c\ln n} \leqslant \ln\ln n, \tag{14}$$

and by virtue of (6) and b) we obtain

$$\ln\frac{n\ln\ln n}{8c\ln n} \geqslant \ln n + \ln\ln\ln n - \ln\ln n - \ln 8 \geqslant \ln n - \ln\ln n. \tag{15}$$

From (12)–(15) we have

$$L^+(F_n) \geqslant \frac{c}{\ln\ln n}\left(\left(n - \frac{n\ln\ln n}{8c\ln n}\right)\left(\ln n - \frac{\ln\ln n}{4c\ln n}\right) + \right.$$

$$\left. + \frac{n\ln\ln n}{8c\ln n}(\ln n - \ln\ln n)\right) + \frac{n\ln\ln n}{8c\ln n} \geqslant \frac{cn\ln n}{\ln\ln n} - \frac{n}{4\ln n} - \frac{n\ln\ln n}{8\ln n} + \frac{n\ln\ln n}{8c\ln n}.$$

Hence, since $c \leqslant \frac{1}{2}$ (see (6)) and by virtue of b), we obtain

$$L^+(F_n) \geqslant \frac{cn\ln n}{\ln\ln n} - \frac{n}{4\ln n} + \frac{n\ln\ln n}{8\ln n} \geqslant \frac{cn\ln n}{\ln\ln n}.$$

The theorem is completely proved.

Lemma 3. $L(F_n) \leqslant 3n - 4$.

Proof. The circuit of Fig. 4 contains $3n - 4$ contacts and realizes the function $F_n(x_1, \dots, x_n)$. Indeed, suppose that this circuit realizes the function f.

Then, on the one hand, every path with nonzero conductivity has the form $x_i\bar{x}_{i+1}\bar{x}_{i+2}\dots\bar{x}_{j-1}x_j$; therefore $f \leqslant F_n$. On the other hand,

$$x_i x_{i+1} \vee x_i\bar{x}_{i+1}x_{i+2} \vee \dots \vee x_i\bar{x}_{i+1}\bar{x}_{i+2}\dots\bar{x}_{j-1}x_j = x_i x_{i+1} \vee x_i x_{i+2} \vee \dots \vee x_i x_j;$$

therefore $f \geqslant F_n$. Thus the lemma is proved.

Theorem 1 follows from Theorem 2 and Lemma 3.

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* This process of forming a branch is similar to the process of forming inessential occurrences of variables in a formula, used in (8).

** From (7) it follows that

$$\frac{\ln\ln n}{8c\ln n} \leqslant \frac{1}{8};$$

moreover,

$$-\ln\left(1-\frac{1}{8}\right)\Big/\frac{1}{8} = 8\ln\frac{8}{7} < 2.$$

Note: Figure translations are in progress. See original paper for figures.