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Abstract

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M. GIMADISLAMOV

ON EXPANSION IN EIGENFUNCTIONS OF A NON-SELF-ADJOINT SYSTEM OF DIFFERENTIAL EQUATIONS ON THE WHOLE AXIS

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In the present paper an expansion in eigenfunctions is obtained for a non-self-adjoint operator generated by a system of second-order differential equations on the whole axis.

We consider a system of differential expressions of the 2nd order, which is written briefly in the form

$$l(y) = -y'' + P(x)y, \tag{1}$$

where $y(x)$ is a k-dimensional column vector; $P(x)$ is a complex-valued matrix of order k.

If $(1+x^2)^{1/2}|P(x)| \in L'_k(-\infty,\infty)$, then the differential expression (1) generates a closed operator L in $L^2_k(-\infty,\infty)$ with domain D, consisting of vector-functions $y(x) \in L^2_k(-\infty,\infty)$ which have an absolutely continuous derivative and $l(y) \in L^2_k(-\infty,\infty)$. In what follows we shall assume that

$$e^{a|x|}|P(x)| \in L'_k(-\infty,\infty). \tag{2}$$

Denote by $s = \sqrt{\lambda}$, $s = \sigma + it$, $0 \le \arg s \le \pi$, and by $Y_1(x,s)$ and $Y_2(x,s)$ linearly independent solutions of the matrix equation

$$-Y'' + P(x)Y = s^2 Y. \tag{3}$$

Under condition (2), these solutions may be chosen so that, for any fixed x, they are analytic in s in the domain $\operatorname{Im} s > -a/2$ and have the following asymptotics in x:

$$Y_1(x,s) = e^{isx}[1 + o(1)] \qquad \text{as } x \to \infty, \tag{4}$$

$$Y_1(x,s) = e^{isx}\left[1 - \int_{-\infty}^{\infty} \frac{e^{-is\xi}}{2is} P(\xi) Y_1(\xi,s)\,d\xi + o(1)\right] \qquad \text{as } x \to -\infty;$$

$$Y_2(x,s) = e^{-isx}[1 + o(1)] \qquad \text{as } x \to -\infty,$$

$$Y_2(x,s) = e^{-isx}\left[1 - \int_{-\infty}^{\infty} \frac{e^{is\xi}}{2is} P(\xi) Y_2(\xi,s)\,d\xi\right] \qquad \text{as } x \to +\infty \tag{5}$$

uniformly in s for $\operatorname{Im} s > -a/2$.

Denote

$$W(s) = \det\begin{pmatrix} Y_1(x,s) & Y_2(x,s) \\ Y_1'(x,s) & Y_2'(x,s) \end{pmatrix}.$$

Theorem 1. *If condition (2) is satisfied, the spectrum of the operator consists of a finite number of eigenvalues—the zeros of the function $W(\lambda^{1/2})$—and of a continuous spectrum on the nonnegative half-axis $\lambda \geqslant 0$.*

Let us note that the point $\lambda = 0$ is not an eigenvalue. In what follows, for simplicity, we shall assume that: 1) $W(\lambda^{1/2})$ has no zeros on the axis σ; 2) the zeros of $W(\lambda^{1/2})$ are simple.

Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the eigenvalues, and let $y_1(x), y_2(x), \dots, y_r(x)$ be the corresponding eigenvector-functions of the operator L. Then the following holds:

Theorem 2. *If conditions 1) and 2) are satisfied and $G(x,\xi,\lambda)$ is the resolvent kernel of the operator L, then for any point λ not belonging to the spectrum of the operator L,*

$$G(x,\xi,\lambda) = \sum_{j=1}^{r} \frac{y_j(x) z_j^*(\xi)}{\lambda_j - \lambda} + \frac{1}{\pi}\int_0^{\infty} \sum_{i=1}^{2} \frac{\Theta_i(x,\sigma) A_i(\sigma) \Psi_i(\xi,\sigma)}{\sigma^2 - \lambda}\,d\sigma, \tag{6}$$

where the integral on the right converges absolutely and uniformly with respect to x, ξ in the interval $-\infty < x, \xi < \infty$. Here $\Theta_i(x,\sigma)$ are linearly independent solutions of the matrix equation (3), and $\Psi_i(x,\sigma)$ are linearly independent solutions of the matrix equation

$$-Z'' + ZP(x) = \sigma^2 Z; \tag{7}$$

$A_i(\sigma)$ are matrices of order k.

We outline the proof of Theorem 2. First we construct the Green' s function $G(x, \xi, \lambda)$ for the operator L, i.e., we solve the equation $l(y) - \lambda y = f$, where $f(x) \in L_k^2(-\infty, \infty)$:

$$G(x, \xi, \lambda) = K(x, \xi, \sqrt{\lambda}) = \frac{1}{W(s)} \begin{cases} -Y_1(x, s) S_{12}(\xi, s), & \text{for } \xi < x, \\ Y_2(x, s) S_{22}(\xi, s), & \text{for } \xi > x, \end{cases} \tag{8}$$

where

$$\frac{1}{W(s)} \begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = 1. \tag{9}$$

Denote by $C_{R,\delta}$ the contour in the s-plane consisting of the straight line $\tau = \delta > 0$ from $\sigma = -\sqrt{R^2 - \delta^2}$ to $\sigma = \sqrt{R^2 - \delta^2}$ and the arc of the circle $s = Re^{i\theta}$ from $\theta = \eta = \arcsin \frac{\delta}{R}$ to $\theta = \pi - \eta$. Choose δ and R_0 so that, if λ_0 is an eigenvalue, then $\operatorname{Im} \lambda_0^{1/2} \neq \delta$ and $R_0^2 > |\lambda_0|$.

Consider the integral

$$I_{R,\delta} = \int_{C_{R,\delta}} \frac{K(x, \xi, s)}{s^2 - \lambda}\, ds \tag{10}$$

for $R \geqslant R_0$; the point $\sqrt{\lambda}$ lies inside the contour $C_{R,\delta}$. Applying the residue theorem and letting $R \to \infty$, we obtain:

$$G(x, \xi, \lambda) = \sum_{j=1}^{r} \frac{G^{(j)}(x, \xi)}{\lambda_j - \lambda} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(\sigma + i\delta) K(x, \xi, \sigma + i\delta)}{(\sigma + i\delta)^2 - \lambda}\, d\sigma, \tag{11}$$

where $G^{(j)}(x, \xi) = y_j(x) z_j^*(\xi)$, and $n(\delta)$ denotes the number of eigenvalues lying inside $C_{R,\delta}$.

By virtue of conditions 1) and 2), as $\delta \to 0$ equality (11) takes the form

$$G(x, \xi, \lambda) = \sum_{j=1}^{r} \frac{y_j(x) z_j^*(\xi)}{\lambda_j - \lambda} + \frac{1}{\pi i} \int_{0}^{\infty} \frac{\sigma K(x, \xi, \sigma) - \sigma K(x, \xi, -\sigma)}{\sigma^2 - \lambda}\, d\sigma. \tag{12}$$

Taking into account relation (9) and the asymptotic formulas (4) and (5), one obtains that:

$$\begin{aligned}
S_{12}(x, s) &= (-1)^k (2is)^{k-1} e^{-isx} [W_1(s) \cdot 1 + o(1)] && \text{as } x \to \infty, \\
S_{12}(x, s) &= (-1)^k (2is)^{k-1} e^{-isx} [1 - A(s) + o(1)] && \text{as } x \to -\infty, \\
S_{22}(x, s) &= (-1)^{k+1} (2is)^{k-1} e^{isx} [1 - B(s) + o(1)] && \text{as } x \to \infty, \\
S_{22}(x, s) &= (-1)^{k+1} (2is)^{k-1} e^{isx} [W_1(s) \cdot 1 + o(1)] && \text{as } x \to -\infty,
\end{aligned}$$

where

$$W_1(s) = \det\left(1 - \int_{-\infty}^{\infty} \frac{e^{-is\xi}}{2is} PY_1\, d\xi\right) = \det\left(1 - \int_{-\infty}^{\infty} \frac{e^{is\xi}}{2is} PY_2\, d\xi\right),$$

$$[W_1(s)]^{k-1} = \det(1 - A(s)) = \det(1 - B(s)).$$

From this asymptotics and from the properties of the Green function it follows that $S_{12}(x,s)$ and $S_{22}(x,s)$ are linearly independent solutions of the matrix equation (7). We choose linearly independent solutions of equation (7) so that

$$S_{12}(x,s) = (-1)^k (2is)^{k-1} [W_1(s)]^{\frac{k-1}{k}} Z_2(x,s),$$

$$S_{22}(x,s) = (-1)^k (2is)^{k-1} [W_1(s)]^{\frac{k-1}{k}} Z_1(x,s),$$

$$\det\begin{pmatrix} Z_2 & Z_1 \\ Z_2' & Z_1' \end{pmatrix} = W(s).$$

Then relation (9) takes the form:

$$\frac{(-1)^k (2is)^{k-1} [W_1(s)]^{\frac{k-1}{k}}}{W(s)} \begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix} \begin{pmatrix} Z_2' & Z_2 \\ Z_1' & Z_1 \end{pmatrix} = 1. \tag{13}$$

Let us compute $\sigma K(x,\xi,\sigma) - \sigma K(x,\xi,-\sigma)$ for $\xi \le x$:

$$\begin{aligned} \sigma K(x,\xi,\sigma) - \sigma K(x,\xi,-\sigma) &= \frac{(-1)^{k+1} \sigma (2i\sigma)^{k-1} [W_1(\sigma)]^{\frac{k-1}{k}}}{W(\sigma)} Y_1(x,\sigma) Z_2(\xi,\sigma) \\ &\quad + \frac{(-1)^k \sigma (-2i\sigma)^{k-1} [W_1(-\sigma)]^{\frac{k-1}{k}}}{W(-\sigma)} Y_1(x,-\sigma) Z_2(\xi,-\sigma) \\ &= \frac{(-1)^{k+1} \sigma (2i\sigma)^{k-1} [W_1(\sigma)]^{\frac{k-1}{k}}}{W(\sigma)} Y_1(x,\sigma) \sum_{i=1}^{k} e_i v_2^{(i)*}(\xi,\sigma) \\ &\quad + \frac{(-1)^k \sigma (-2i\sigma)^{k-1} [W_1(-\sigma)]^{\frac{k-1}{k}}}{W(-\sigma)} \sum_{i=1}^{k} y_1^{(i)}(x,-\sigma) e_i^* Z_2(\xi,-\sigma), \end{aligned} \tag{14}$$

where e_i is the unit column vector; e_i^* is the unit row vector; $v_2^{(i)*}(\xi,\sigma)$ is the i-th row of the matrix $Z_2(\xi,\sigma)$; $y_1^{(i)}(x,-\sigma)$ is the i-th column of the matrix $Y_1(x,-\sigma)$.

Since $Z_1(\xi,-\sigma)$ and $Z_2(\xi,-\sigma)$ are fundamental matrices of (7), it follows that

$$v_2^{(i)*}(\xi,\sigma) = a_1^{(i)*} Z_1(\xi,-\sigma) + a_2^{(i)*} Z_2(\xi,-\sigma), \tag{15}$$

where $a_1^{(i)*}, a_2^{(i)*}$ are constant row vectors.

We determine the row vectors $a_1^{(i)*}$ and $a_2^{(i)*}$ as follows: we differentiate equality (15); the equality obtained, together with (15), gives the following system of equations:

$$v_2^{(i)*}(\xi,\sigma) = a_1^{(i)*}Z_1(\xi,-\sigma) + a_2^{(i)*}Z_2(\xi,-\sigma),$$
$$v_2^{(i)*\prime}(\xi,\sigma) = a_1^{(i)*}Z_1'(\xi,-\sigma) + a_2^{(i)*}Z_2'(\xi,-\sigma).$$

Taking (13) into account, from this system we obtain

$$a_1^{(i)*} = \frac{(-1)^k(-2i\sigma)^{k-1}[W_1(-\sigma)]^{\frac{k-1}{k}}}{W(-\sigma)}\,[v_2^{(i)*}(\xi,\sigma)Y_2'(\xi,-\sigma) + v_2^{(i)*\prime}(\xi,\sigma)Y_2(\xi,-\sigma)],$$

$$a_2^{(i)*} = \frac{(-1)^k(-2i\sigma)^{k-1}[W_1(-\sigma)]^{\frac{k-1}{k}}}{W(-\sigma)}\,[v_2^{(i)*}(\xi,\sigma)Y_1'(\xi,-\sigma) + v_2^{(i)*\prime}(\xi,\sigma)Y_1(\xi,-\sigma)].$$

Similarly, for the column vector $y_1^{(i)}(x,-\sigma)$ we have

$$y_1^{(i)}(x,-\sigma) = Y_1(x,\sigma)b_1^{(i)} + Y_2(x,\sigma)b_2^{(i)}, \tag{16}$$

where $b_1^{(i)}, b_2^{(i)}$ are constant column vectors, which we determine analogously to the preceding ones, and we obtain:

$$b_1^{(i)} = \frac{(-1)^k(2i\sigma)^{k-1}[W_1(\sigma)]^{\frac{k-1}{k}}}{W(\sigma)}\,[Z_2'(x,\sigma)y_1^{(i)}(x,-\sigma) + Z_2(x,\sigma)y_1^{(i)\prime}(x,-\sigma)],$$

$$b_2^{(i)} = \frac{(-1)^k(2i\sigma)^{k-1}[W_1(\sigma)]^{\frac{k-1}{k}}}{W(\sigma)}\,[Z_1'(x,\sigma)y_1^{(i)}(x,-\sigma) + Z_1(x,\sigma)y_2^{(i)\prime}(x,-\sigma)].$$

Substituting the expressions found for $v_2^{(i)*}(\xi,\sigma)$ and $y_1^{(i)}(x,-\sigma)$ into (14), we obtain that, for $\xi \leqslant x$,

$$\sigma K(x,\xi,\sigma) - \sigma K(x,\xi,-\sigma) =$$

$$= \frac{\sigma(2i\sigma)^{k-1}(-2i\sigma)^{k-1}[W_1(\sigma)W_1(-\sigma)]^{\frac{k-1}{k}}}{W(\sigma)W_1(-\sigma)}\sum_{j=1}^{2} Y_j(x,\sigma)T_j(\sigma)Z_j(\xi,-\sigma), \tag{17}$$

where

$$T_2(\sigma) = Z_2(x,\sigma)Y_2'(x,-\sigma) + Z_2'(x,\sigma)Y_2(x,-\sigma),$$

$$T_1(\sigma) = -[Z_1(x,\sigma)Y_1'(x,-\sigma) + Z_1'(x,\sigma)Y_1(x,-\sigma)].$$

In a similar way we arrive at the same expression (17) also for $\xi > x$. Denoting

$$\Theta_j(x,\sigma) = \frac{\sigma(2i\sigma)^{k-1}[W_1(\sigma)]^{\frac{k-1}{k}}}{W(\sigma)} Y_j(x,\sigma),$$

$$\Psi_j(\xi,\sigma) = \frac{\sigma(-2i\sigma)^{k-1}[W_1(-\sigma)]^{\frac{k-1}{k}}}{W(-\sigma)} Z_j(\xi,-\sigma),$$

$$A_i(\sigma) = \frac{1}{i\sigma}T_i(\sigma),$$

we obtain formula (6).

From the expansion (6) it is not difficult to obtain an expansion for any vector-function $y(x) \in D_L$ and an analogue of Parseval' s equality.

It is likewise not difficult to write the expansion for $G(x,\xi,\lambda)$ when the zeros of $W(s)$ are multiple.

Remark 1. If $W(s)$ has zeros on the real axis $\tau = 0$, then the expansion for $G(x,\xi,\lambda)$ is obtained by the method of paper (1).

Remark 2. If $(1+x^2)^{1/2}|P(x)| \in L_k'(-\infty,\infty)$, then the expansion (6) can be obtained if $W(s)$ has no zeros on the real axis.

Remark 3. For $k = 1$, from the expansion (6) one can obtain Kemp' s result (3).

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