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Abstract

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CLASSES AND SETS OF SUBGROUPS OF FINITE GROUPS

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In a number of works devoted to the existence of subgroups in finite groups, special attention is given to the question of the existence of classes or sets (1) of subgroups possessing one or, simultaneously, several specified properties. The well-known Sylow theorem indicates the existence in finite groups of classes or sets of special (p-subgroups (2)) subgroups. There are numerous works whose subject of investigation is the existence of classes of nonspecial subgroups (2,3), as well as their sets (1). A number of authors (4–14) have devoted their investigations to groups with a prescribed number of classes of noninvariant or nonattainable (15) subgroups. In these works one usually considers classes of conjugate, isomorphic, or isoordic (11) subgroups.

In the present work all these investigations are continued in the direction of finding a larger number of classes and sets of subgroups, as well as additional properties of these subgroups. For example, it is proved that a nonsolvable group has τ (τ is the number of distinct prime divisors of the order of the group) isoordic nonattainable subgroups which are simultaneously nonspecial and solvable, and the order of each of them is divisible by no more than three distinct prime numbers.

From the results obtained there follow the principal results of (7,9–14), as well as the result of (3) on the number of nonspecial subgroups in finite groups. Theorem 5 of the present paper is a strengthening of our previously published (16) results. In addition, the paper contains a solution of the supposition expressed in (14) that for every n one can indicate such an r_0 that all groups of π-type n and π-rank exceeding r_0 will be special.

In what follows the following notation will be used: G is a finite group of order $(G) = g$; E is the identity group; the identity group will be included among special, solvable, and also primary (see below) groups; π is some nonempty set of prime numbers; n_π is the largest π-divisor of the natural number n; $\pi(n)$ is the set of distinct prime divisors of the number n; $\tau(G) = \tau$ is the number of distinct prime divisors of the order of the group G; $t_\pi(G) = t$ is the number of distinct prime divisors of the order of the group G that belong to π; a πd-group (πd-subgroup) is a group (subgroup) whose order is divisible by at least one prime number from π; Γ_G is the set of all proper subgroups of the group G; $\Gamma_G^{\pi d}$ is the set of all πd-subgroups from Γ_G; Γ_G^s is the set of all solvable subgroups from Γ_G; $\Gamma_G^{\bar{n}}$ is the set of all nonspecial subgroups from Γ_G; $\Gamma_G^{\bar{n}i}$ is the set of all nonattainable subgroups from Γ_G; $\Gamma_G^{(k)}$ is the set of all subgroups from Γ_G whose orders are divisible by no more than k (k is a natural number) distinct prime numbers. By a Hall subgroup we shall mean a subgroup whose order and index are relatively prime; by a primary subgroup, a subgroup whose order is a power of a prime number. Introduce a single-valued function f,

the domain of definition of which is some (empty or not) subset of the set Γ_G, and the range is some subset of the set π. The function f must satisfy the following two conditions: 1) the function f has one and the same value on isomorphic subgroups; 2) to the value p ($p \in \pi$) of the function f there corresponds, as argument, a pd-subgroup.

The fact that the function f on some set Γ assumes all possible values from π will be written as follows: $f(\Gamma) = \pi$. If the function f is defined on the whole set $\Gamma_G^{\pi d}$, then, in view of the single-valuedness of this function, it partitions the set of all proper πd-subgroups of the group G into disjoint classes. Each class of isomorphic πd-subgroups of the group G belongs entirely to one of the classes obtained by means of the function f. For different functions we shall use the notations f and φ.

Let now θ denote some group-theoretic property. Then a group (subgroup) possessing the property θ will be called a θ-group (θ-subgroup). We shall consider only such properties θ for which the product of invariant θ-subgroups is again a θ-subgroup. By the property θ one may mean being special, solvability, π-separability, being a π-subgroup, etc.

By a maximal attainable θ-subgroup of a group G we shall mean a proper attainable θ-subgroup which is not a proper subgroup of any other attainable proper θ-subgroup of the group G.

We present the results obtained.

Theorem 1. *If in a finite group G there exists a true attainable θ-subgroup, then in G there is also a true invariant θ-subgroup. A maximal attainable θ-subgroup of the group G is invariant in G.*

Theorem 2. *Let H be an invariant subgroup of order h of a group G, and suppose there exists a function φ such that*

$$\varphi\left(\Gamma_H^{\pi d} \cap \Gamma_H^{s} \cap \Gamma_H^{\bar{n}} \cap \Gamma_H^{(3)}\right) = \pi(h_\pi).$$

In order that there exist a function f such that

$$f\left(\Gamma_G^{\pi d} \cap \Gamma_G^{s} \cap \Gamma_G^{\bar{n}} \cap \Gamma_G^{(3)}\right) = \pi(g_\pi),$$

it is sufficient that the set $\Gamma_H^{\bar{n}}$ not be empty.

Theorem 3. *If G is not p-nilpotent* $^{(1)}$ *for every $p \in \pi(g_\pi)$, then there exists a function f such that*

$$f\left(\Gamma_G^{\pi d} \cap \Gamma_G^{\bar{n}} \cap \Gamma_G^{(2)}\right) = \pi(g_\pi).$$

The exception is formed by p-special $^{(1)}$ *groups of type S when $\pi(g_\pi) = \{p\}$.*

Theorem 4. *If G is a nonsolvable group, then the set*

$$\Gamma_G^{\bar{n}} \cap \Gamma_G^{\bar{n}i} \cap \Gamma_G^{(2)}$$

contains at least three classes of isomorphic subgroups.

Theorem 5. *If the group G is nonsolvable, then there is a function f such that*

$$f\left(\Gamma_G^{s} \cap \Gamma_G^{\bar{n}} \cap \Gamma_G^{\bar{n}i} \cap \Gamma_G^{(3)}\right) = \pi(g).$$

Theorem 6. *A nonsolvable group contains no fewer than $2\tau + 1$ classes of isomorphic unattainable solvable subgroups, the order of each of which is divisible by no more than three distinct primes; moreover, among these classes there exist 2τ classes which can be partitioned into pairs in such a way that to each prime divisor p of the order of the group there correspond two classes of pd-subgroups.*

Theorem 7. *If G is a nonspecial E_π-group* $^{(17)}$ *and its S_π-subgroup* $^{(18)}$ *contains no fewer than 2^{t-2}, $t > 1$, classes of isomorphic solvable*

Hall subgroups, then G has no fewer than 2^{t-2} classes of isomorphic irreducible solvable Hall πd-subgroups.

We note that solvable, π-solvable, π-separable groups, as well as groups possessing the property E_π (see $(^{18})$), satisfy the condition of Theorem 7.

Let us now denote by r the number of classes of isomorphic solvable noninvariant πd-subgroups of the group G.

Theorem 8. Let $n = t - r - 2$, $r_0' = 2n - 4$, and let r_0'' be the integer part of the root of the equation $x = 2^{x-n}$. Then, for $r > \max\{r_0', r_0''\}$, the group G is special.

Theorem 8 is also a solution of the conjecture contained in the paper $(^{14})$.

Theorem 9. Let

$$G = H_1 \times H_2 \times \cdots \times H_n, \quad n \geqslant 1,$$

where H_i $(i = 1, 2, \ldots, n)$ are Hall nonprimary subgroups of the group G, indecomposable into a direct product of proper Hall subgroups, and suppose that among $H_1, H_2, \ldots, H_n$ there are k $(k \geqslant 0)$ solvable π-subgroups. Then G has at least one πS-set $(^{1})$ containing no fewer than $t - k$ subgroups.

Since in Theorem 9 k cannot exceed $\left[\frac{t}{2}\right]$, this theorem is a refinement of Theorem 4 of S. A. Chunikhin from the paper (^{1}).

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Note: Figure translations are in progress. See original paper for figures.

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