M-SETS AND HAUSDORFF MEASURE

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Abstract

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O. S. IVASHEV-MUSATOV

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In 1916 D. E. Menshov constructed the first example of an M-set with Lebesgue measure equal to zero, i.e., of such a set $\mathcal{E}$ of Lebesgue measure zero for which there exists a trigonometric series

$$\sum_{n=-\infty}^{+\infty} c_n e^{inx} \qquad (c_{-n} = \bar{c}_n),$$

not all of whose coefficients are equal to zero, and which converges to zero outside $\mathcal{E}$. However, for the set $\mathcal{E}$ constructed by him the p-dimensional Hausdorff measure $\mu_p\mathcal{E} = +\infty$ for every $p < 1$. Subsequently, various authors (for more details see ([1])) constructed many examples of an M-set with Lebesgue measure equal to zero. In this connection, for a number of constructions it is easy to verify that, for any p, $0 < p < 1$, the parameters of the construction can be chosen so that the resulting M-set will have p-dimensional Hausdorff measure equal to zero. Let us note that, in the construction of M-sets, the arithmetic nature of the parameters of the constructed set usually plays an important role, and often also the symmetry of the set itself. In this note an example is constructed of a perfect M-set $\mathcal{E}$, whose p-dimensional Hausdorff measure $\mu_p\mathcal{E} = 0$ for every $p > 0$. At the same time, the arithmetic nature of the set itself, as is clear from the construction, is completely immaterial, and the set itself is highly "asymmetric."

Theorem. *There exists a perfect M-set $\mathcal{E}$ for which the p-dimensional Hausdorff measure $\mu_p\mathcal{E} = 0$ for every $p > 0$.*

The construction of this set is carried out by the usual method ([1]), indicated already by D. E. Menshov,–one has to construct a perfect set $\mathcal{E} \subset [0, 2\pi]$ with $\mu_p\mathcal{E} = 0$ for every $p > 0$, and a single-valued continuous function $F(x)$ on $[0, 2\pi]$, constant on each interval contiguous to the perfect set $\mathcal{E}$, and such that its Fourier–Stieltjes transform

$$\Phi(y) = \int_0^{2\pi} e^{-iyx}\, dF(x)$$

tends to zero as $y \to \infty$. Then, as is known ([1]), the trigonometric series

$$\sum_{n=-\infty}^{+\infty} c_n e^{inx}, \quad \text{where } c_n = \int_0^{2\pi} e^{-inx}\, dF(x),$$

converges to zero outside $\mathcal{E}$, and consequently $\mathcal{E}$ is an M-set.

It is easy to see that it suffices to prove the following lemma:

Lemma. *On $[0, 2\pi]$ there exists a continuous monotone nondecreasing function $u(x)$, constant (but not identically constant on $[0, 2\pi]$) on each interval contiguous to some perfect set E with $\mu_p E = 0$ for every $p > 0$, and such that*

$$\lim_{y\to\infty} \int_0^{2\pi} e^{-iyx}\, du(x) = 0. \tag{1}$$

Construction of the set E. Choose a sequence of positive numbers ε_m, tending to zero monotonically decreasing.

1st step of the construction. We divide the interval $[0, 2\pi] = \Delta_{0,0}$ into $m_1 = n_1$ equal intervals $\delta_{1,j} = [b_{1,j-1}, b_{1,j}]$, $j = 1, 2, \dots, n_1$. Choose a real number $p_{1,0}$ so that the inequality $m_1^{1-\varepsilon_1}/p_{1,0}^{\varepsilon_1} < 3/4$ is satisfied. Denote by $\Delta_{1,j} = [\alpha_{1,j}, \beta_{1,j}]$, $j = 1, \dots, n_1$, the interval of length $\delta_{1,j}/p_{1,0}$ lying on the interval $\delta_{1,j}$.

2nd step of the construction. Choose an integer $m_2 > 4p_{1,0}m_1$ and a sequence of monotonically increasing integers $Q_{2,j}$, $j = 1, 2, \dots, n_1$, $Q_{2,1} = 1$, $Q_{2,j} > 4jp_{1,0}$. Each interval $\Delta_{1,j}$ of the first step of the construction is divided into $m_2 Q_{2,j}$ equal intervals $\delta_{2,l} = [b_{2,l-1}, b_{2,l}]$, numbered consecutively from left to right

$$\left(l = 1, 2, \dots, n_2 = m_2 \sum_{j=1}^{n_1} Q_{2,j}\right).$$

Choose a sequence of real numbers $p_{2,j}$, $j = 1, 2, \dots, n_1$, so that the inequalities $(m_2 Q_{2,j})^{1-\varepsilon_2}/p_{2,j}^{\varepsilon_2} < 3/4$ are satisfied, and denote by $\Delta_{2,l} = [\alpha_{2,l}, \beta_{2,l}]$, $l = 1, 2, \dots, n_2$, the interval of length $\delta_{2,l}/p_{2,j}$ lying on the interval $\delta_{2,l}$.

k-th step. Suppose that we have already carried out the first $k-1$ steps of the construction. This means that the numbers $m_s, n_s, Q_{s,i}, p_{s,i}$ $(i = 1, \dots, n_{s-1})$ and the intervals $\delta_{s,j}, \Delta_{s,j}$ $(j = 1, \dots, n_s)$ have already been chosen for $s = 1, 2, \dots, k-1$. Choose also a sequence of numbers $P_{k-1,j}$ $(j = 1, \dots, n_{k-1})$ as follows. For each interval $\Delta_{k-1,j}$ there is uniquely determined an interval Δ_{s,i_s} containing it, for every $s = 1, 2, \dots, k-2$. Put

$$(c_{j0} = 0) \quad P_{k-1,j} = \prod_{s=1}^{k-1} p_{s,i_{s-1}}.$$

We note immediately that all $P_{k-1,j}$ are equal to one another for all indices j for which the intervals $\Delta_{k-1,j}$ lie on one and the same interval $\Delta_{k-2,i}$, and, moreover, $P_{k-1,j} = P_{k-2,i}p_{k-1,i}$. Then the k-th step of the construction of the set E consists in the following. Choose an integer m_k so that the inequalities

$$m_k > n_{k-1}, \qquad m_k > 2^k m_{k-1}, \qquad m_k > 2k \sum_{j=1}^{n_{k-1}} P_{k-1,j} \tag{2}$$

are satisfied, and divide the interval $\Delta_{k-1,1}$ into $m_{kQ_{k,1}}$, where $Q_{k,1} = 1$, equal intervals $\delta_{k,l} = [b_{k,l-1}, b_{k,l}]$, $l = 1, 2, \dots, m_{kQ_{k,1}} = l(1)$. Choose a real number $p_{k,1}$ so that the inequality $(m_{kQ_{k,1}})^{1-\varepsilon_k}/p_{k,1}^{\varepsilon_k} < 3/4$ is satisfied, and denote by $\Delta_{k,l} = [\alpha_{k,l}, \beta_{k,l}]$, $l = 1, 2, \dots, l(1)$, the interval of length $\delta_{k,l}/p_{k,1}$ lying on the interval $\delta_{k,l}$. Define, in addition, the numbers $P_{k,l} = P_{k-1,1}p_{k,1}$, $l = 1, 2, \dots, l(1)$. Suppose that we have already divided the intervals $\Delta_{k-1,s}$, $s = 1, 2, \dots, j-1$, into smaller intervals $\Delta_{k,l}$, $l = 1, 2, \dots, l(j-1)$, and, consequently, have determined the numbers $Q_{k,s}, p_{k,s}$ and $P_{k,l} = P_{k-1,s}p_{k,s}$. Then choose an integer $Q_{k,j}$ so that

$$Q_{k,j} > Q_{k,j-1}, \qquad m_{kQ_{k,j}} > 2k \sum_{l=1}^{l(j-1)} P_{k,l}, \tag{3}$$

and divide the interval $\Delta_{k-1,j}$ into $m_{kQ_{k,j}}$ equal intervals $\delta_{k,l} = [b_{k,l-1}, b_{k,l}]$, $l = l(j-1)+1, \dots, l(j)$. Choose a real number $p_{k,j}$ so that the inequality

$$\frac{(m_{kQ_{k,j}})^{1-\varepsilon_k}}{p_{k,j}^{\varepsilon_k}} < \frac{3}{4}, \tag{4}$$

is satisfied.

and denote by $\Delta_{k,l} = [\alpha_{k,l}, \beta_{k,l}]$, $l = l(j-1)+1, \dots, l(j)$, the interval of length $\delta_{k,l}/p_{k,j}$ lying on the interval $\delta_{k,l}$. In addition, put

$$P_{k,l} = P_{k-1,j}p_{k,j}. \tag{5}$$

Thus, the intervals $\Delta_{k-1,j}$, $j = 1, 2, \dots, n_{k-1}$, of the $(k-1)$-st step of the construction will contain smaller intervals $\Delta_{k,l}$, $l = 1, 2, \dots, n_k =$

$$= m_k \sum_{j=1}^{n_{k-1}} Q_{k,j},$$

of the k-th step of the construction. It is easy to see that the indices j and $l(j)$ indicated above are such that $b_{k-1,j} = b_{k,l(j)}$. Let us also note the relations between the lengths of the intervals $\Delta_{k-1,j} \supset \delta_{k,l} \supset \Delta_{k,l}$:

$$\delta_{k,l} = \frac{\Delta_{k-1,j}}{m_k Q_{k,j}}, \qquad \Delta_{k,l} = \frac{\delta_{k,l}}{p_{k,j}} = \frac{\Delta_{k-1,j}}{m_k Q_{k,j} p_{k,j}} = \frac{2\pi}{P_{k,l} \prod_{s=1}^{k} m_s Q_{s,j}}. \tag{6}$$

Put $E = \bigcap_{k=1}^{\infty} E_k$, where $E_k = \bigcup_{l=1}^{n_k} \Delta_{k,l}$. From the construction it is clear that $E_{k-1} \supset E_k$ and that E is a perfect set.

We shall show that the p-dimensional Hausdorff measure of the constructed set E is equal to zero for every $p > 0$. Fix p and define k_0 so that $p \geq \varepsilon_{k_0}$. Estimate $\mu_p E_k$ for $k > k_0 + 1$:

$$\mu_p E_k = \sum_{l=1}^{n_k} \Delta_{k,l}^p = \sum_{j=1}^{n_{k-1}} \sum_{l=l(j-1)+1}^{l(j)} \Delta_{k,l}^p = \sum_{j=1}^{n_{k-1}} \left(\frac{\Delta_{k-1,j}}{m_{kQ_{k,j}} p_{k,j}} \right)^p m_{kQ_{k,j}} =$$

$$= \sum_{j=1}^{n_{k-1}} \frac{(m_{kQ_{k,j}})^{1-p}}{p_{k,j}^p} \Delta_{k-1,j}^p < \sum_{j=1}^{n_{k-1}} \frac{(m_{kQ_{k,j}})^{1-\varepsilon_k}}{p_{k,j}^{\varepsilon_k}} \Delta_{k-1,j}^p < \frac{3}{4} \mu_p E_{k-1},$$

whence it follows that $\mu_p E = 0$ for any $p > 0$.

Construction of the function $u(x)$**.** Define on the interval $[0, 2\pi]$ the sequence of functions

$$\varphi_k(x) = \begin{cases} P_{k,l}, & x \in \Delta_{k,l}, \\ 0, & x \in \overline{E_k}, \end{cases} \tag{7}$$

and the sequence of monotonically nondecreasing continuous functions

$$u_k(x) = \int_0^x \varphi_k(t)\, dt, \tag{8}$$

for which

$$u_k(0) = 0, \qquad u_k(2\pi) = 2\pi. \tag{9}$$

The first equality is obvious, and the second follows from the fact that, by (7), (5), (6) and the construction of the set E ($\Delta_{k-1,j} \supset \delta_{k,l} \supset \Delta_{k,l}$),

$$\int_{\delta_{k,l}} \varphi_k(x)\, dx = P_{k,l} \Delta_{k,l} = P_{k-1,j} p_{k,j} \Delta_{k,l} = P_{k-1,j} \delta_{k,l} = \int_{\delta_{k,l}} \varphi_{k-1}(x)\, dx,$$

whence it is easily obtained that

$$u_k(2\pi) = u_{k-1}(2\pi),$$

$$u_1(2\pi) = \int_0^{2\pi} \varphi_1(x)\,dx = \sum_{j=1}^{n_1} p_{1,0}\Delta_{1,j} = p_{1,0} n_1 \frac{\Delta_{0,0}}{n_1 p_{1,0}} = \Delta_{0,0} = 2\pi.$$

The sequence of functions $\{u_k(x)\}$ converges uniformly on the interval $[0, 2\pi]$. Indeed, consider the graph of the difference

$$z = u_{k-1}(x) - u_k(x) = \int_0^x [\varphi_{k-1}(t) - \varphi_k(t)]\,dt.$$

It is clear that this is a piecewise-linear function. From Fig. 1 it is seen that, by virtue of (6) and (2),

$$|u_{k-1}(x) - u_k(x)| < \max_l P_{k,l}\Delta_{k,l} = \max_j \frac{2\pi}{\prod_{s=1}^k m_s Q_{s,j}} < \frac{1}{2^k}.$$

Then

$$|u_m(x) - u_{m+r}(x)| \le \sum_{\nu=m+1}^{m+r} |u_{\nu-1}(x) - u_\nu(x)| < \sum_{\nu=m+1}^{m+r} \frac{1}{2^\nu} < \frac{1}{2^m},$$

whence uniform convergence on the interval $[0, 2\pi]$ follows. Thus, on the interval $[0, 2\pi]$ there exists a limiting function

$$u(x) = \lim_{k\to\infty} u_k(x),$$

continuous, monotone nondecreasing, not identically constant by virtue of (9), and constant on every contiguous interval of the perfect set E. The latter is clear from the fact that each function $u_k(x)$ is constant on every contiguous interval of the set E_ν, for any $\nu = 1, 2, \dots, k$.

Fig. 1

The proof of relation (1) follows the same pattern as the proof of the analogous relation in the author’ s paper (2). In making the estimate it is convenient to use the form of the graph of the difference $u_{k-1}(x) - u_k(x)$, shown in Fig. 1.

Similarly, the following is proved.

Theorem. *For every monotone function $h(u) > 0$, $h(u) \to 0$ as $u \to 0$, there exists an M-set with h-measure equal to zero.*

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Note: Figure translations are in progress. See original paper for figures.